

SIMULATION AND CONTROL
OF DISTRIBUTED PARAMETER SYSTEMS

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Simulation and Control of Distributed Parameter Systems

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ABSTRACT

The present work is an attempt to put together the most relevant aspects of the engineering problems involving distributed parameter systems (D.P.S.'s). Simulation and optimal control are explained in detail in Chapters II and III.

The original contribution of this thesis is given in Chapters V and VI, where modal control theory and a gradient subroutine that searches for the optimal reference coefficients are used. As a result, it was possible to obtain an output distribution better than the one achievable by the known methods. This technique works in situations of strongly nonlinear control and compensates the effect of having the analyzer and synthesizer approximated by low order matrices. It also makes it possible to give higher weight to some zones of the output distribution in order to have a better local fit. The necessary background for understanding Chapters V and VI is given in Chapter IV.

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LIST OF SYMBOLS AND ABBREVIATIONS¹

A	Cross section
\tilde{A}	System matrix
B	Control operator
\tilde{B}	Control matrix
C	Heat capacity (II), total capacitance (III)
C_i	Constant
\tilde{C}	Output vector
c	Capacitance per unit length (III), specific heat (V)
D.P.	Distributed parameter
D.P.S.	Distributed parameter system
F	Heat flow
\tilde{F}	Vector of feedback multipliers
$f(\cdot)$	Function of a variable
$G(s)$	Transfer function
H	Hamiltonian
\tilde{H}	Auxiliary transformation
$H_{eq}(s)$	Feedback multiplier
H_i	Distribution of the i^{th} manipulator
h_{ij}	Generalized Fourier coefficients of H_i
I.C.	Initial condition
J	Cost functional
K	Kernel (III), thermal conductivity (III)

¹ The Roman Numbers inside parenthesis stand for the chapters where the symbols have the indicated meaning.

\tilde{K}	Solution of matrix Riccati equation
k_i	Shifting in the i^{th} eigenvalue
k	Time integration interval (II), Number of control points (III)
L	Distance boundary (III), direct operator (IV, V)
$L[\cdot]$	Laplace transform operator
M	Upper bound (III), inverse operator (IV, V)
M^+	Adjoint operator of M
$m(x,t)$	Forcing function
N	Order of the approximated system
n	Discrete index
\hat{P}	Solution of Equation 3.62
p_j	Lagrange multiplier
$p(x,t)$	Pressor
Q	Weighting operator
q	System's temperature (III), heat flow (V)
R	Total resistance
\tilde{R}	Weighting matrix
r	Resistance per unit length (III), number of manipulators (IV)
S	Lateral area per unit length
\tilde{S}	Weighting matrix
s	Number of sensors
T	Time boundary (III), superscript transpose (III-IV), temperature (I, III)
t_f	Final time
t	Time
U	Temperature
u	Eigenfunction (IV, V), temperature (II), control function (III)

\underline{V}	Vector of first n eigenfunctions
\underline{v}	Fluid velocity (III), temperature of the medium (III), eigenfunction of adjoint operator (IV, V)
\underline{W}	Vector of coefficients of eigenfunction expansion (III), close loop matrix (IV, V)
x	Cartesian coordinate (abscissa)
\underline{x}	State vector
y	Cartesian coordinate (ordinate)
$y(x,t)$	Temperature
z	Complex variable
α_i	Coefficient
$\alpha_i(t)$	Actual input to the i^{th} manipulator
Φ	Feedback multiplier
Λ	Diagonal matrix
λ_i	i^{th} eigenvalue
$\mu_i(t)$	Generalized Fourier coefficient of $m(x,t)$
$\mu_i(s)$	Generalized Fourier coefficient of $m(x,s)$
$\omega_i(t)$	Generalized Fourier coefficient of $y(x,t)$
$\omega_i(s)$	Generalized Fourier coefficient of $y(x,s)$
τ_i	Generalized Fourier coefficient of desired output
$\delta[\cdot]$	Small variation
ρ	Mass density
\exists	Such that
$\ \cdot\ $	Norm of
$ \cdot $	Absolute value of
\sim	When underlining a letter means matrix or vector
\langle, \rangle	Inner product
\downarrow	Pseudo inverse

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I also want to acknowledge my wife and children for the hundreds of hours I should have dedicated to them, but the continuous work on this thesis together with other scholar studies did not enable me to do so.

I. GENERALITIES ABOUT THE CONTROL OF DISTRIBUTED PARAMETER SYSTEMS

A. HISTORY

Distributed parameter systems have existed in Nature since the beginning, and many of these systems were naturally stable.

For example, during the period of pre-living beings dramatic changes in temperature originated no less dramatic variations of the lithosphere. These earth's convolutions, together with many subsidiary effects, tried then to make changes in different places until a practical equilibrium was obtained. What was this, other than a distributed parameter system?

With the advent of life in this planet other D.P.S.'s appeared, such as the nitrogen cycle, in which different forms of life receive nitrogen and give it up to other forms which, in turn, originate new nitrogen.

In the beginning of this century much attention was dedicated to the mathematical description of D.P.S.'s such as transmission lines. With the entrance on the scene of modern control theory and integrated circuits concepts, some attempts were made toward a generalization of those ideas, but practically until the 1950's not much had been realized. By that time a certain number of systems with time delays could be analyzed using what are called conventional control techniques, but the scope of these techniques was quite limited.

In the early 1960's some technical papers were published in an attempt to apply the concepts of optimal control to D.P.S.'s and, from then on, a highly geometric rate of technical paper production took place.

A new kind of approach to the control of D.P.S.'s was carried on by Murray-Lasso [Ref. 45] and Gould [Ref. 26] in 1965, using the generalization of the concepts of modal control introduced by Rosenbrock [Ref. 53]. This approach is treated in depth in Chapters IV, V and VI. References 62 and 18 also contain very useful information in this field.

In 1968 the Int. Journal of Control published an extensive bibliography by Wang [Ref. 72], one of the scientists in the U.S.A. who has contributed most significantly to the development of the D.P.S.'s theory. In November 1969, the Aerospace Research Laboratories published the "Survey of optimal Control of Distributed Parameter Systems," by A. C. Robinson [Ref. 52], which contains 261 references and an analysis of various ways of implementing the optimal control of such systems. Special reference ought to be made to the contributions of Lions [Ref. 40] and Butkovskiy [Ref. 5], which respectively in France and U.S.S.R. established foundations of a scientific treatment of the optimal control of D.P.S.'s.

In 1969 and 1970 numerous technical papers on D.P.S.'s were published. Many of them may be found in the "Proceedings of the Joint Automatic Control Conference," "Int. Journal of Control" and "Simulation."

Almost all the above references, although necessary, are very theoretical and not much has been done by the control engineer in order to implement those methods or to describe them in an easy language. It is a purpose of this thesis to fill a little of this existing gap.

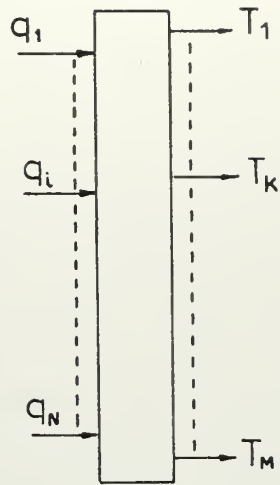
History is not only the study of the past events but also the careful analysis of them in order to extrapolate for the future. It seems reasonable, on the basis of past history to make some predictions as to future studies of D.P.S.'s.

- i) From a technological point of view it is a known fact that the impact of control science in the improvements accomplished in the last two decades constituted a "sine qua non" factor for these trends. At the actual rates of development of new D.P.S.'s (as mentioned before the integrated circuits are a characteristic example) and of the capability to describe and control them, it seems that within a few years the control science will be applicable to almost every different aspect of the existing technology.
- ii) The actual and very important ecological problems, namely water and air pollution, if mathematically described will assume the form of partial differential equations. The consequences of being able to deal with such problems in a scientific way can be easily implied.
- iii) It seems realizable that partial differential equations may be derived that approximate the behavior of some segments of society, at least during certain periods of

time. When such equations are found it should be possible to use computer optimization (parameter identification) to determine values or expressions for the coefficients. These coefficients are but the historical constants the historians have been looking for.

- iv) Man is a complex being, and his free nature leads to impulsive behavior at random intervals. The effect of such behavior on society is often observable. It is reasonable to think that the phenomena of man's behavior may in some way follow a Gaussian distribution, as do many natural phenomena. If equations can be obtained to represent society as a D.P.S., then it may be possible to treat the impulsive aspects of man's behavior as Gaussian noise and perhaps the consequences of such behavior can be minimized by building into the social system something similar to a Kalman filter.
- v) As progress is made in developing equations for various sub-systems within the social systems, it may well be that some sub-systems may be found uncontrollable, others controllable. Then the theory of D.P.S.'s may well become an important tool for redesign of social systems.

Finally, in order to give a geometrical interpretation of the behavior of a D.P.S., Figs. 1.1 and 1.2 are included, which show how each single input contributes to the value of the output at every point.



q_i - The i^{th} control
is a heat input.

T_j - The j^{th} output,
i.e. the temperature.

Figure 1.1. A D.P.S.: Heating of a Rod

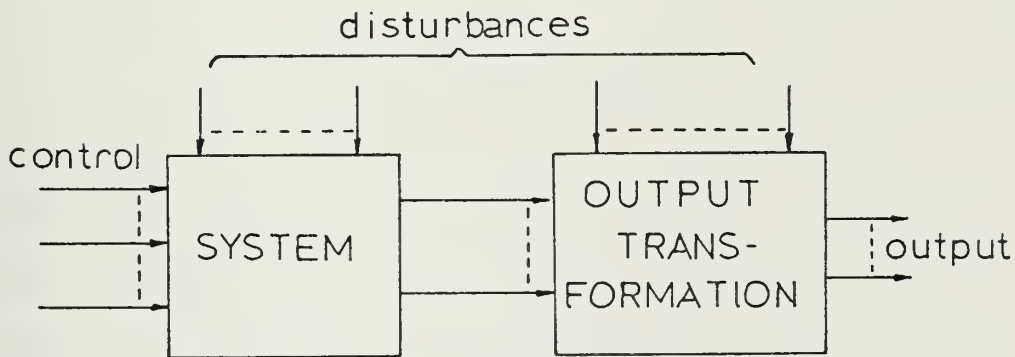


Figure 1.2. A More General Model

Note: The output transformation uses the values measured by some existent sensors and converts them into the desired output.

B. HOW TO CONTROL DISTRIBUTED PARAMETER SYSTEMS

The ordinary differential equations used by the control engineer to describe lumped parameter systems, with time as independent variable, did not prove sufficiently accurate to deal with complex systems where either the input, or the output, or both, are also function of other parameters, generally of spatial nature. Therefore, a new theory was developed in order to extend some of the ideas so well understood when dealing with ordinary differential equations to the much more difficult problems described by partial differential equations and integral equations.

There are four basic approaches to the control problem of D.P.S.'s; however, none of them is a general method applicable to all types of these systems:

1. Use of Time-Delay Techniques

The use of time-delay techniques [Ref. 64] has been common when dealing with problems such as those that arise in long pneumatic control lines, heat transfer in nuclear plants, production delay in assembly lines and in general to every kind of control systems where time-delays are involved. The conventional analysis and design techniques applied to these systems requires too much work and for this reason they are now studied almost entirely by computer simulation.

2. Reduction of the Partial Differential Equations to Ordinary Differential or Difference Equations

Once the ordinary differential or difference equations are obtained lumped parameter techniques must be used. From a

control point of view the most representative methods used to obtain the above equations are as follows:

a. Space Quantization (DSCT)

The space derivatives are replaced by finite differences but the time derivatives are maintained. One of the biggest advantages of this method is that it permits an analog modeling of the system. The accuracy can be improved by increasing the number of sections, which must be adequately isolated. When the number of sections becomes too large for the required accuracy, another method must be chosen.

b. Time and Space Quantization (DSDT)

Use is made of the well-known techniques for the numerical solution of partial differential equations. Although quite valuable from an analytic point of view and for simulation, it seems a little lengthy if one is trying to apply it to control design.

c. Laplace Transform

The Laplace transform method uses the fact that $L[\frac{\partial f(x,t)}{\partial x}] = \frac{\partial F(x,s)}{\partial x}$ and $L[\frac{\partial F(x,t)}{\partial t}] = sF(x,s) - f(x,0^+)$, whenever $f(x,t)$ is transformable, $L[f(x,t)] = F(x,s)$ and $\frac{\partial f(x,t)}{\partial x}$ exists.

The problems that can be treated by Laplace transformations are quite numerous, the principal objection being the intricate expressions to which one arrives when dealing with systems somehow more complex (i.e. multidimensional systems).

3. Modal Control

The modal control approach, Refs. 26 and 45, consists in replacing the partial differential operator with an infinite order matrix which is approximated by another matrix of order $N < \infty$. Through adequate matrix manipulations it is possible to uncouple the interconnected inputs and outputs, such that separate loops are established, each of which can be treated by lumped parameter control techniques.

Its most noteworthy advantage is the great insight that it can give to a problem. The range of operation is, however, limited because it can be applied only to completely continuous, bounded and linear or linearized operators.² Also, sometimes, the truncation is problematic, either because some of the higher order eigenvalues may be of fundamental importance or because the convergence of these eigenvalues may be too slow. In this case the system would require a high order matrix representation, which is not very acceptable in practice. The last drawback seems to be the difficulty that often arises in the computation of the eigenvalues and eigenfunctions. A detailed analysis of this theory is given in Chapter IV.

4. Optimal Control

Optimal control is, by far, the technique about which the largest amount of work has been published. This comes from the fact that the control of systems described by partial differential equations varies remarkably with changes in the initial and boundary conditions, with the definition of cost

² See Appendix B for definitions.

function and from system to system. Some more or less generalized solutions have been proposed, normally according to the classification of the system operator within the three following categories: elliptic, parabolic or hyperbolic partial differential operator. Optimal control will be discussed in a certain depth in Chapter III; it seems to be the approach of greatest future. This is due to the increasing knowledge on how to describe the cost function, giving convenient weight to each of its components, and also due to the tendency for the control systems to become highly computerized because of the low cost of integrated circuitry.

The four basic approaches just described are not completely independent of each other. Effectively, in many circumstances, all of them are ultimately reduced to the solution of problems involving ordinary differential or difference equations. There is, however, an important reason for the chosen criterion, which can be better explained giving the following example. Consider para. 2; it is obvious that after the transformation is accomplished the lumped parameter optimal control techniques can be used; the difference from para. 5 is in the way the optimal control is derived. In the first case the optimal control law is derived from an ordinary differential or difference equation, while in the last one it is derived directly from the given partial differential equation.

II. SIMULATION METHODS FOR DISTRIBUTED PARAMETER SYSTEMS

A. INTRODUCTION

The distributed parameter part of a physical system is most accurately described by partial differential equations. Analysis and design of such systems requires solution of the partial differential equations. Solution by analytical methods can be quite laborious, and such solutions are not helpful in design problems; indeed they may not be convenient for many system analysis problems.

An alternate approach is the simulation of the distributed parameter system in a computer (analog, digital or hybrid) and the use of this simulation model in computer studies planned for analysis and/or design. A number of different simulation techniques have been developed for this purpose. Some of the methods have been chosen to fit a specific type of computing facility, others have been designed to solve a specific class of partial differential equations, and still others are general methods that can be applied to a variety of problems.

The purpose of this chapter is to classify and list most of the available simulation methods, giving also some of the important facts associated with each of them. No detailed comparison or evaluation is attempted since the choice of a method is guided largely by available computer facilities, the nature of the specific problem to be solved, the accuracy needed in the solution, and the constraints on cost and time.

However the chapter is concluded with a summarizing table which indicates some of the pertinent considerations associated with each method. This table could be used as a start of a critical comparison and evaluation.

B. A GENERALIZED CLASSIFICATION

It can be said that, in general, the partial differential equation describing a distributed parameter system is part of an overall control system. In order to analyze the behavior of this complete system it is necessary to know how to solve the corresponding p.d.e. This can be done either by analytical methods or by simulation.

Simulation has the great advantage that once the system is represented for a specific input and initial conditions any change of conditions is easily realizable; such is not the case for the analytical computation.

The techniques for dealing with multidimensional systems are only extensions of those used for unidimensional ones. Due to the fact that the simulation of those systems involves an enormous amount of hardware, if implemented by analog methods, most multidimensional simulations have been done in the digital computer and sometimes in the hybrid computer. They are, however, tremendously time consuming and for this reason only the two-dimensional systems can be modeled with reasonable, if not great accuracy [Ref. 49]. The objective of a simulation will be to have for specific positions the output as a function of time or, for each instant of time, the output

as a function of the coordinates. In a two-dimensional system the solution is normally obtained keeping two coordinates fixed, changing the other one and repeating this for several combinations of points and time instants until a complete grid is obtained. Actually some oscilloscopes are already capable of providing multidimensional pictures. This thesis will deal only with one-dimensional systems, for which the following methods of solution may be considered:

DSDT	-	discrete-space, discrete-time
DSCT	-	discrete-space, continuous-time
CSDT	-	continuous-space, discrete-time
TSCT	-	transformed-space, continuous-time
TSDT	-	transformed-space, discrete-time

Schuchmann [Ref. 59] gives a brief and clear analysis of these methods, involving error and stability considerations, as well as the relative advantages of each one. He doesn't consider the three possibilities resulting from taking the time Laplace transform, explaining that this would imply the need for getting its inverse; this, although feasible is very time consuming. However, when considering the solution at few points (sometimes only one) the technique seems to be good, in particular if using infinite product expansions [Ref. 22]. Such technique gives the solution as a product of terms (truncated infinite series) in "s", each of them easily implementable in an analog computer; it will be discussed in more detail in Section F of this chapter.

1. DSDT - Discrete-Space, Discrete-Time

With the very fast digital computers actually existent and also because of their enormous accuracy, popularity

and availability, most of the work has the tendency to be done by the DSDT method. Also with the actual video-display units it is possible to get CRT pictures and change easily the parameters without need for an analog or hybrid computer.

The basic concepts of the discretization are very simple and, because of its importance, all of the next section will be devoted to this problem.

2. DSCT - Discrete-Space, Continuous-Time

This is the method generally used when there is available only the analog computer and the transformation techniques described previously in this chapter are difficult to implement. If the number of nonlinearities or the required accuracy is large, the lack of enough multipliers or other components in the computer does not permit solution.³ To avoid such situation the multiplex method has been tried; it consists in switching on and off sequentially, with digital signals, the parallel operation of the circuits corresponding to each of the discretized positions and in using the same analog hardware to simulate the nonlinearities of each branch of the parallel combination.

In Fig. 2.1 is shown how the heat equation

$$C \frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left(k \frac{\partial u}{\partial x} \right) \quad (2.1)$$

³ Some companies work with huge analog computers, as for example English Electronics Co. which operates the 1,500-amplifier Saturn computer.

can be simulated in the analog computer. C is the heat capacity and the heat flow F is given by

$$F = -k \frac{\partial u}{\partial x} \quad (2.2)$$

If the temperature " u " is measured at integer stations and the flux at half-integer stations, Eq. 2.2 may be written in

discrete form as $F_{n-1/2} = -k_{n-1/2} \frac{u_n - u_{n-1}}{\Delta x}$ and Eq. 2.1 will

$$\text{become } \frac{\partial}{\partial x} \left(k \frac{\partial u}{\partial x} \right) \Big|_n = - \frac{\partial F}{\partial x} \Big|_n = - \frac{F_{n+1/2} - F_{n-1/2}}{\Delta x}$$

from which

$$C_n \frac{du_n}{dt} = - \frac{F_{n+1/2} - F_{n-1/2}}{\Delta x} \quad (2.3)$$

The above procedure may be easily implemented according to the schematic diagram on the following page.

3. CSDT - Continuous-Space, Discrete-Time

This technique is advisable for the simulation of the flow equation

$$\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} = f(x, t)$$

which, in discrete form, is written as

$$\frac{du_n(x)}{dx} = - \frac{1}{v \Delta t} [u_n(x) - u_{n-1}(x)] + \frac{f_n(x)}{v} \quad (2.4)$$

From this equation it is obvious that Δt can be decreased without increasing the number of components, the only limitation being the stability requirement ($1/v\Delta t < 1$). The analog computer with adequate number of track-hold units or the hybrid computer are the most efficient ways of solution of this problem.

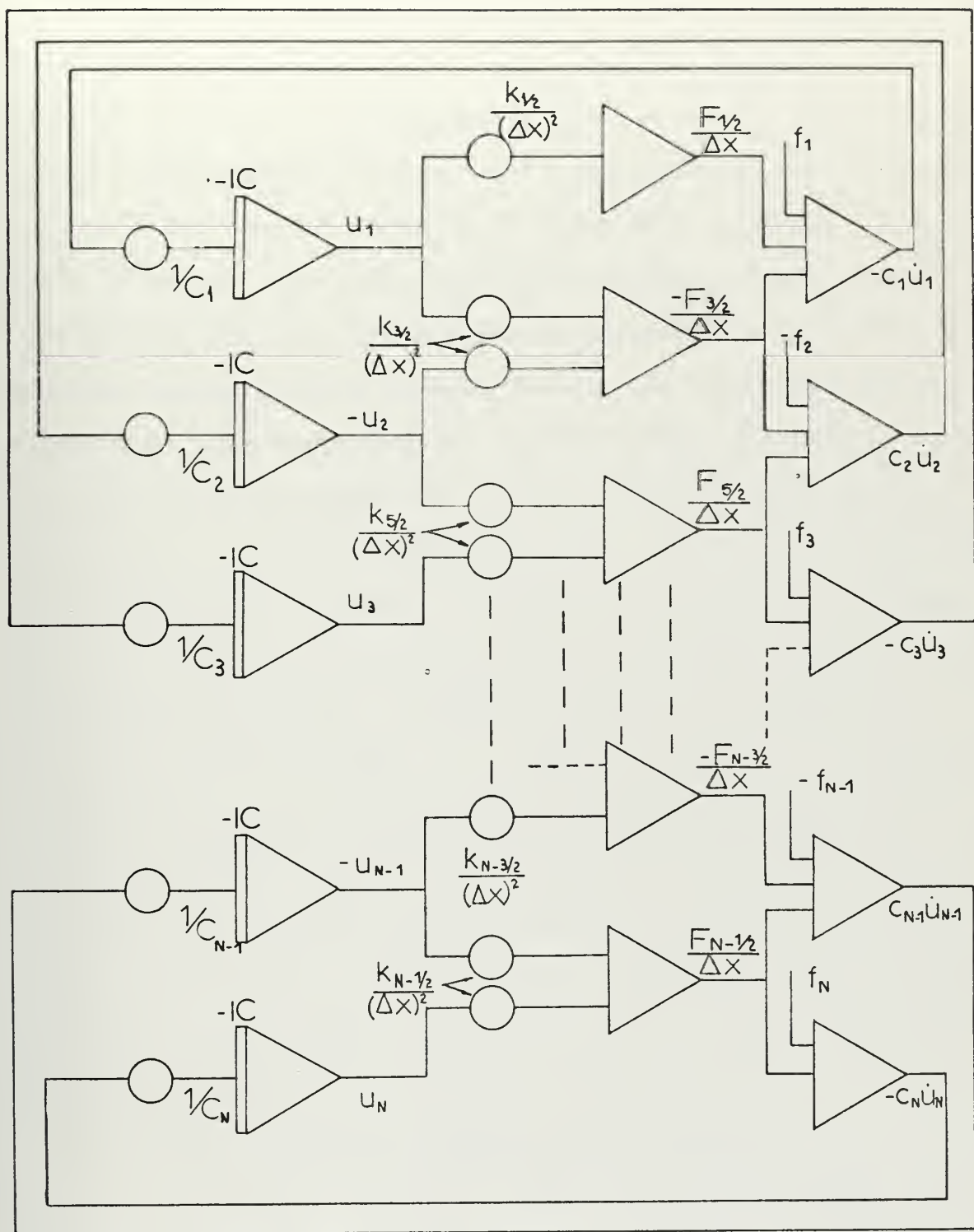


Figure 2.1. Analog Computer Simulation of the One Dimensional Heat Equation

4. TSCT - Transformed-Space, Continuous-Time

TSCT is a method applicable to systems represented by self-adjoint operators (i.e. heat and wave equations) and, through convenient assumptions also to non-self adjoint systems. The concepts involved are intimately related with those of the modal control which will be studied in depth in Chapters IV and V. The most remarkable example of this method is the Bubnov-Galerkin transformation described in Appendix A. Another representative example is the one developed and explained in very comprehensive terms by Hong and Larson [Ref. 31] which can be applied to any set of first-order partial differential equations in normal form. It gives the solution as a continuous function of distance for any value of time. The only trade-off is the requirement for the computation of the inverse transform. However, because the output is $U(t,s)=f_0-f_1s+1/2!(f_2s^2)-\dots$, where f_n ($n=0,1,2,\dots$) are exclusively functions of time, the inverse transform is very easily obtained. In the example given in the paper the analog solution of a single nonlinear first-order p.d.e. required only 2 integrators, 3 multipliers, 5 amplifiers and 1 divider. Because of the excellent features of this method a more detailed analysis will be given in Section E.

5. TSDT - Transformed-Space Discrete-Time

TSDT is an extension of the above method and it is becoming increasingly more popular. One example of such a method is implemented in the computer run No. 7 of this thesis.

C. DIGITAL COMPUTER SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS (DSDT)

The principal source of information for this section is the work, "Numerical Solution of Partial Differential Equations," by G. D. Smith [Ref. 63], an excellent book written in very comprehensive terms. For sophisticated problems more advanced texts are recommended, such as Refs. 1 and 17. Due to their relevancy only second-order p.d.e.'s will be considered.

1. Parabolic Equation⁴

Consider the parabolic p.d.e. $\frac{\partial U}{\partial T} = K \frac{\partial^2 U}{\partial x^2}$ which normalized using the transformation $t = kT/L^2$, $x = \frac{X}{L}$ and $u = \frac{U - U_0}{U_0 - U_0}$, where U_0 is some given value of U (generally maximum or minimum) at zero time. The resulting equation is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad (2.5)$$

Written in discrete form the above equation becomes

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2} \quad (2.6)$$

⁴ Given the second order quasi-linear p.d.e.

$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + e = 0$ where a, b, c, e may not be functions of second or higher order derivatives (according to the definition of quasi-linearity), if $b^2 - 4ac$ is equal to zero the equation is said to be parabolic, if greater than zero is hyperbolic and is elliptic when less than zero.

or

$$u_{i,j+1} = u_{i,j} + r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \quad (2.7)$$

Geometrically the interpretation of this formula is simple, as can be seen from Fig. 2.2.

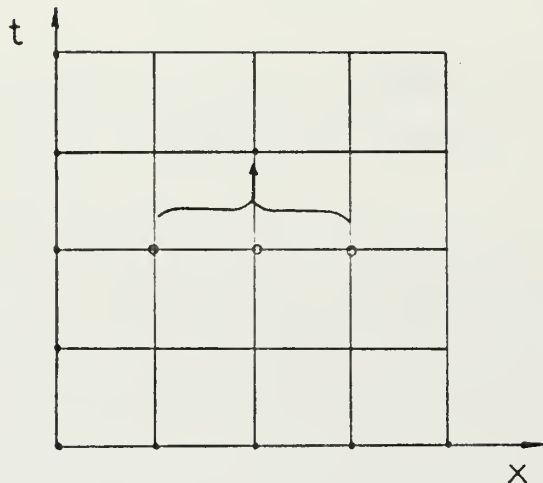


Figure 2.2 Grid Solution of a Parabolic Differential Equation

Here, all the values at $t=0$ and $x=0$ or l are known. The computation is, therefore, straightforward and does not require iterations. In the case of mixed boundary conditions of the type $au + b\frac{\partial u}{\partial n} = c$, where n is the outward normal to the surface and a, b, c are functions of the space coordinates and time, only the values of u at $t=0$ are known. In this case, the solution of the problem becomes only slightly more difficult, since it is necessary to evaluate with a simple

formula the boundary at $x=0$ for all discrete times, at the start of the computation of the elements of each row [Ref. 47].

This may be done introducing a new difference equation based on the boundary condition, as for example

$\frac{u_{i,j} - u_{0,j}}{\delta x} = h(u_{0,j} - v)$ or $\frac{u_{i,j} - u_{-1,j}}{2\delta x} = h(u_{0,j} - v)$, which is more accurate.

The last equation has $u_{-1,j}$ as unknown and therefore there is need for still another one assuming that the heat equation is satisfied at $x=0$.

The explicit method, although the simplest one, has the inconvenient of only being convergent for $0 \leq k/h^2 \leq 1/2$. For this reason some other methods were developed which are unconditionally convergent for all values of $r=k/h^2$.

The most popular one is the Crank-Nicolson implicit method, which is represented by the equation:

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{1}{2} \left\{ \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} \right\} \quad (2.8)$$

or

$$-ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + (2-2r)u_{i,j} + ru_{i+1,j} \quad (2.9)$$

and assuming that the system is discretized in $N-1$ sections, for each instant of time, $N-1$ simultaneous equations must be solved. The best way of solving this system of equations is using any of the well-known iterative techniques applied to

the digital computer. The best known techniques are Jacobi, Gauss-Seidel and successive over-relaxation (s.o.r.). The first one is very slow and for this reason, in general, only the other two are used, rewriting Eq. 2.9 as

$$u_{i,j+1} = u_{i,j} + \frac{1}{2}r\{(u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}) + (u_{i-1,j} - 2u_{i,j} + u_{i+1,j})\} \quad (2.10)$$

and dropping the index $j+1$, Eq. 2.10 may now be expressed as

$$u_i = \frac{1}{2}r(u_{i-1} - 2u_i + u_{i+1}) + b_i \quad (2.11)$$

$$\text{where } b_i = u_{i,j} + \frac{1}{2}r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \quad (2.12)$$

According to the above notations and also denoting the iterations by superscripts, the Gauss-Seidel scheme is obtained:

$$u_i^{(n+1)} = \frac{r}{2(1+r)}\{u_{i-1}^{(n+1)} + u_{i+1}^{(n)}\} + \frac{b_i}{1+r} \quad (2.13)$$

It converges for all $r > 0$. In matrix form, making $\rho = \frac{r}{2(1+r)}$

$$\begin{bmatrix} u_1^{(n+1)} \\ u_2^{(n+1)} \\ u_3^{(n+1)} \\ \vdots \\ u_{N-1}^{(n+1)} \end{bmatrix} = \begin{bmatrix} 0 & \rho & 0 & \dots \\ 0 & \rho^2 & \rho & \dots \\ 0 & \rho^3 & \rho^2 & \rho & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \rho^{N-1} & \rho^{N-2} & \dots \end{bmatrix} \begin{bmatrix} u_1^{(n)} \\ u_2^{(n)} \\ u_3^{(n)} \\ \vdots \\ u_{N-1}^{(n)} \end{bmatrix} + \underline{\underline{C}} \quad (2.14)$$

The successive over-relaxation scheme is the fastest one and is obtained directly from Eq. 2.13 by adding and subtracting $u_i^{(n)}$ to the right-hand side.

The following expression is obtained:

$$u_i^{(n+1)} = u_i^{(n)} + \omega \left[\frac{r}{2(1+r)}\{u_{i-1}^{(n+1)} + u_{i+1}^{(n)}\} + \frac{b_i}{(1+r)} - u_i^{(n)} \right] \quad (2.15)$$

where ω is the relaxation factor (usually $1 < \omega < 2$).

For maximum rate of convergence

$$\omega_b = \frac{2}{1 + \sqrt{(1 - \mu^2)}} \quad \text{where } \mu = \frac{r}{1+r} \cos \frac{\pi}{N} \quad (2.16)$$

2. Hyperbolic Equations

The most commonly used method for the numerical solution of hyperbolic equations is the method of "characteristics" and it consists in integrating the given p.d.e. along chosen directions (when dealing with two-dimensional systems; if three dimensional systems are considered the integration is in chosen planes) for which the partial derivatives are reduced to ordinary derivatives.

The mathematical explanation is as follows: given the quasi-linear equation

$$a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial^2 u}{\partial x \partial y} + c \frac{\partial^2 u}{\partial y^2} + e = 0 \quad (2.17)$$

with $b^2 - 4ac > 0$ and defining $\frac{\partial u}{\partial x} = p, \frac{\partial u}{\partial y} = q, \frac{\partial^2 u}{\partial x^2} = r, \frac{\partial^2 u}{\partial x \partial y} = s$ and $\frac{\partial^2 u}{\partial y^2} = t$, can now be written:

$$dp = \frac{\partial p}{\partial x} dx + \frac{\partial p}{\partial y} dy = r dx + s dy$$

and

$$dq = \frac{\partial q}{\partial x} dx + \frac{\partial q}{\partial y} dy = s dx + t dy \quad (2.18)$$

Also from Eq. 2.17: $ar + bs + ct + e = 0$, which is used together with Eq. 2.18 to eliminate r and t , giving the final expression

$$s \left\{ a \left(\frac{dy}{dx} \right)^2 - b \left(\frac{dy}{dx} \right) + c \right\} - \left\{ a \frac{dp}{dx} \frac{dy}{dx} + c \frac{dq}{dx} + e \frac{dy}{dx} \right\} = 0 \quad (2.19)$$

Choosing $\frac{dy}{dx}$ such that the polynomial inside the left { } is zero, it will be necessary to solve only

$$a \frac{dp}{dx} \frac{dy}{dx} + c \frac{dq}{dx} + e \frac{dy}{dx} = 0 \quad (2.20)$$

where $\frac{dy}{dx}$ is known and takes two real values.

A detailed procedure for the computer implementation of this technique is given in Ref. 63, pp. 101-102.

The method of characteristics is the most accurate one for solving hyperbolic p.d.e.'s and whenever discontinuities are involved is the one that must be used. However, if no discontinuities are present, a convergent finite differences method is much easier to implement and will give generally adequate results. One scheme that may be used when dealing with the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ is

$$u_{i,j+1} = r^2 u_{i-1,j} + 2(1-r^2)u_{i,j} + r^2 u_{i+1,j} - u_{i,j-1} \quad (2.21)$$

with $r = \frac{\delta x}{\delta t}$, which is convergent for $r \leq 1$.

3. Elliptic Equations

Laplace's and Poisson's equations are the two best known equations of the elliptical type. These equations have the peculiarity of being always integrable in a closed area (if a two-dimensional equation) or volume (if a three-dimensional equation) on which boundaries the function or its gradient are known.

For example Poisson's equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(x,y) \quad (2.22)$$

may be represented in normalized form making $x = \frac{X}{L}$, $y = \frac{Y}{L}$, and $u = \frac{U}{L^2}$. One possible discrete-version may be

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = h^2 f(x_i, y_i) \quad (2.23)$$

which must be verified for every point inside the boundary.

In general, if there are N points inside the boundary a set of N equations will result, linear or nonlinear, in the same way as the original p.d.e.

When the boundaries are curvilinear and the function values or its derivatives cannot be represented accurately by the chosen mesh, either a finer mesh must be used or the following equation:

$$\frac{2u_A}{\theta_1(1+\theta_1)} + \frac{2u_B}{\theta_2(1+\theta_2)} + \frac{2u_3}{1+\theta_1} + \frac{2u_4}{1+\theta_2} - 2\left(\frac{1}{\theta_1} + \frac{1}{\theta_2}\right)u_0 = h^2 f_0 \quad (2.24)$$

according to Fig. 2.3, where f_0 is the value of f at $(0,0)$.

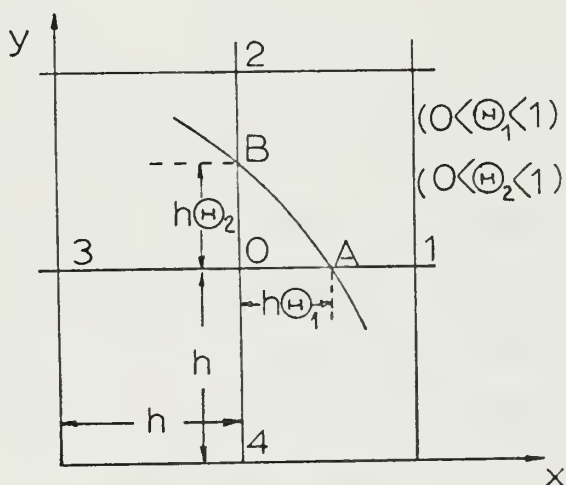


Figure 2.3. Grid for an Elliptic p.d.e. with Curvilinear Boundaries

D. APPLICATION OF THE CONCEPT OF "CHARACTERISTICS" IN HYBRID COMPUTERS (DSCT OR CSDT)

There exists many different ways of solving p.d.e.'s in a hybrid computer. Maybe the most representative of these procedures is the extension of the concept of the characteristics described in the last section. Such an extension was originated by Vichnevetsky and Associates [Refs. 69 and 70]. It is explained as follows: given the p.d.e.

$$\frac{\partial u}{\partial t} + f(x,t) \frac{\partial u}{\partial x} = g(u,x,t) \quad (2.25)$$

with given initial and boundary conditions and making $\frac{dx}{dt} = f(x,t)$ it is possible to write

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial u}{\partial t} + f \frac{du}{dx} \quad (2.26)$$

Therefore

$$\frac{du}{dt} = g(u,x,t) \quad (2.27)$$

which is an ordinary differential equation, integrable by analog computer.

For the boundary condition $u(0,t)$, Eq. 2.27 can be integrated at high speed while the integration of \dot{x} is done slowly (characteristic line). Geometrically, for L the length of the system this method is exemplified in Fig. 2.4. If the highest integration speed is much greater than the lower one the time can be taken as a constant for each fast sweep; otherwise a correction can be implemented. For further information a specific case is worked out with great detail in Ref. 70.

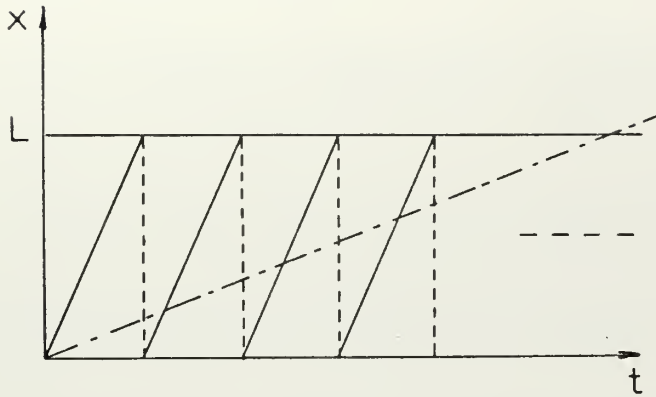


Figure 2.4. Geometrical Representation of the Method of Characteristics in Hybrid Computer

E. ANALOG SIMULATION OF UNIDIMENSIONAL D.P.S.'s BY LAPLACE TRANSFORMING THE SPACE COORDINATES (TSCT)

Following the same procedure as used by Hong and Larson in Ref. 31, consider the system defined by

$$\frac{\partial u}{\partial t} + f(t) \frac{\partial u}{\partial x} + g(t)u = 0 \quad (2.28)$$

with initial and boundary conditions

$$\begin{aligned} u(0,x) &= ae^{-g(0)x} \\ u(t,0) &= a \end{aligned} \quad (2.29)$$

Normalizing the length and by defining $u(t,1)=b$ the distance Laplace transform of u will be

$$U(t,s) = \int_0^1 e^{-sx} u dx = \int_0^1 e^{-sx} u dx \quad (2.30)$$

and, after substituting in Eq. 2.28 it comes out⁵

$$\frac{dU(t,s)}{dt} + f(t) \left[\int_0^{\infty} e^{-sx} \frac{dU(t,s)}{dx} dx \right] + g(t) U(s) = 0 \quad (2.31)$$

or, because the integral is identical to zero for x greater than 1:

$$\frac{dU(t,s)}{dt} + (g(t) + sf(t)) U(t,s) = f(t) [a - e^{-s}b] \quad (2.32)$$

In order to obtain the I.C.'s for the solution of Eq. 2.32 make

$$\begin{aligned} U(t,0) &= \int_0^{\infty} e^{-sx} u(0,x) dx = \int_0^1 e^{-sx} e^{-g(0)x} dx \\ &= \frac{1}{s+g(0)} [1 - e^{-(s+g(0))}] \end{aligned} \quad (2.33)$$

which expanded in McLaurins's series becomes

$$U(t,0) = 1 - \frac{s+g(0)}{2!} + \frac{(s+g(0))^2}{3!} - \frac{(s+g(0))^3}{4!} + \frac{(s+g(0))^4}{5!} - \dots \quad (2.34)$$

or

$$U(t,0) = h_0(g(0)) - h_1(g(1))s + h_2(g(2))s^2 \dots \quad (2.35)$$

⁵ Remember that $L_x \left[\frac{\partial u}{\partial t} \right] = \int_0^{\infty} e^{-sx} \frac{\partial u}{\partial t} dx = \frac{d}{dt} U(t,s)$

$$\text{and } L_x \frac{\partial u}{\partial x} = \int_0^{\infty} e^{-sx} \frac{\partial u}{\partial x} dx = \lim_{P \rightarrow \infty} \int_0^{\infty} e^{-sx} \frac{\partial u}{\partial x} dx$$

$$= \lim_{P \rightarrow \infty} \left\{ e^{-sx} u \Big|_0^P + s \int_0^P e^{-sx} u dx \right\}$$

$$= s \int_0^{\infty} e^{-sx} u dx - u(t,0)$$

$$= sU(t,s) - u(t,0)$$

Equation 2.32 for the derived I.C.'s is known to have a series solution of the form

$$U(t,s) = \sum_{n=0}^{\infty} \frac{(1)^n}{n!} h_n s^n = h_0 - h_1 s + \frac{h_2}{2!} s^2 - \dots \quad (2.36)$$

where h_n ($n=0,1,2,\dots$) depends only on time.

Substituting Eq. 2.36 in Eq. 2.32 and equating the coefficients of equal powers of s provides one set of ordinary differential equations:

$$\begin{aligned} \frac{dh_0}{dt} + gh_0 &= f(a-b) \\ \frac{dh_1}{dt} + gh_1 &= f(h_0-b) \\ \frac{dh_n}{dt} + gh_n &= f(nh_{n-1}-b) \end{aligned} \quad (2.37)$$

The I.C.'s for this set are derived setting $s=0$ in Eq. 2.36 and solving as indicated for the case of $h_1(0)$:

$$\begin{aligned} h_1(0) &= \frac{h_0(0)}{s} - \frac{u_0}{s} \\ &= \frac{(1 - \frac{g_0}{2!} + \frac{g_0^2}{3!} - \dots) - (1 - \frac{s}{2!} - \frac{g_0}{2!} + \frac{s^2 + 2g_0 s + g_0^2}{3!} - \dots)}{s} - \dots \end{aligned} \quad (2.38)$$

The following set is obtained:

$$\begin{aligned} h_0(0) &= 1 - \frac{g_0}{2!} + \frac{g_0^2}{3!} - \frac{g_0^3}{4!} + \dots \\ h_1(0) &= \frac{1}{2!} - \frac{2g_0}{3!} + \frac{3g_0^2}{4!} - \frac{4g_0^3}{5!} + \dots \end{aligned} \quad (2.39)$$

Hong and Larson prove that h_n may be obtained from the moment expression $h_n = \int_0^1 x^n u \, dx$.

The set of $n+1$ Eqs. 2.37 has $n+2$ unknowns and, for this reason, another relationship is required to define a unique solution.

The approach followed by the authors of the paper is to consider the first moment

$$\begin{aligned}
 h_0 &= \int_0^1 u \, dx = xu \Big|_0^1 - \int_0^1 x \frac{\partial u}{\partial x} \, dx \\
 &= u(t,1) - \int_0^1 x \frac{\partial u}{\partial x} \, dx \\
 &= b - \int_0^1 x \frac{\partial u}{\partial x} \, dx
 \end{aligned} \tag{2.40}$$

Given the continuous nature of x , Eq. 2.40 may be written, according to the mean value theorem as:

$$h_0 = b - x \int_0^1 \frac{\partial u}{\partial x} \, dx = b - x(b-a) \tag{2.41}$$

where

$$x = \frac{\int_0^1 xu \, dx}{\int_0^1 u \, dx} = \frac{h_1}{h_0} \tag{2.42}$$

The function \underline{b} is obtained from the above equations and it is given by

$$b = \frac{h_0^2 - ah_1}{h_0 - h_1} \tag{2.43}$$

Equations 2.37, 2.39 and 2.43 are the only requirements for an analog solution of the problem.

The example worked out on the reference paper considers only two equations of the set and even so shows good results. As stated in Section II-B, the important point on this

simulation is the enormous reduction of hardware it affords. This method, if implemented digitally, leads to the TSDT approach.

F. SIMULATION BY INFINITE PRODUCT EXPANSIONS (DSTT)

This technique is based on the Mittag-Leffler theorem which states that a function analytic at the origin and with an infinite number of simple zeros, may be decomposed in one infinite product of simple factors.

Considering the transcendental function $\cos z$, it is equivalent to the product $(1-2z/\pi)(1+2z/\pi)(1-2z/3\pi)(1+2z/3\pi)\dots$, or, if in Taylor's series form, to the sequence $1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots$. The first representation is characterized by the exact preservation of the zeros.

Goodson [Ref. 21] works out some examples for different types of p.d.e.'s. He considers first the equations in the normal form and takes the time Laplace transform in order to obtain an ordinary differential equation (with s as parameter) in state variable form. Once obtained the general solution for the given linear boundary conditions, the infinite product expansion is applied to the transcendental functions in the solution. The comparison between the product expansion and the eigenvalue or Fourier expansions as indicated in Figs. 2.5a and 2.5b, allows the statements:

- (i) Both expansions have the same eigenvalues.
- (ii) The product expansion preserves the extremum values of the exact solution (no general proof but it has been

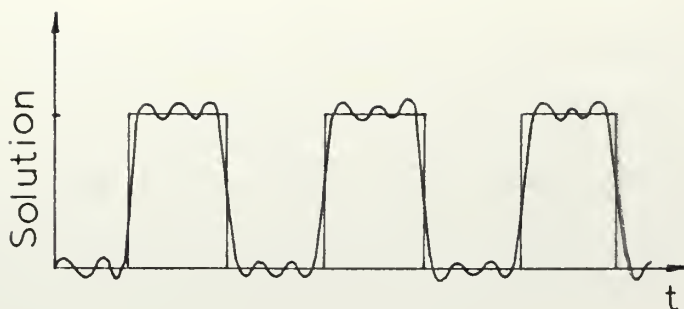


Figure 2.5a. Eigenfunction Expansion

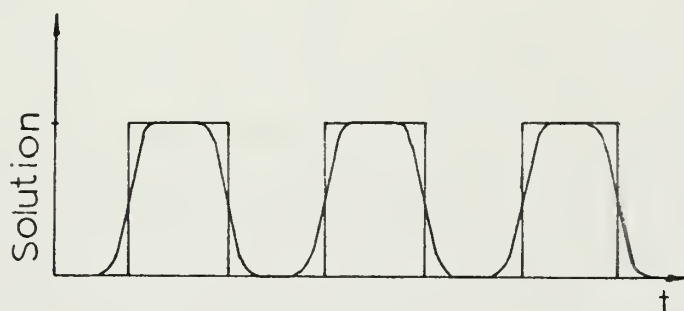


Figure 2.5b. Infinite Product Expansion

verified in several cases), while the Fourier expansion minimizes the mean square error.

- (iii). The product expansion is very suitable for analog simulation due to the nature of its terms. As an example, the solution of the one dimensional heat transfer equation with pointwise control at $x=0$, will yield an expression as follows:

$$u(1,s) = F(s) \prod_{n=1}^{\infty} \frac{1}{1+s/k_n^2}, \text{ where } u(1,s) \text{ is the}$$

Laplace transform of the temperature at $x=1$ and $F(s)$ and k_n are obtained from the data.

(iv) The poles of the transfer function are preserved in the case of the product expansion and, therefore, the technique is recommendable for the simulation of DPS's represented by low-order uni-dimensional (otherwise the analytic computations would become too long) p.d.e.'s.

G. MONTE-CARLO SIMULATION

Although the study of stochastic models is a little beyond the purpose of this thesis, its actual importance is such that some considerations are supposedly worthwhile. Two basic papers were published, Refs. 10 and 41, the first one describing an analog computer approach and the last one a hybrid computer solution. For a good understanding of these papers it is necessary that the reader be familiar with the stochastic simulation of systems described by ordinary differential equations and also with the theory of Markoff processes. An adequate theory of both these fields may be found in Refs. 38, 39 and 48.

The great advantages of this method is that it requires only a very limited amount of analog hardware or digital memory and that it is only necessary to make the computation at the points where the solution is desired.

H. OTHER SIMULATION TECHNIQUES

Many other distributed parameter systems have been simulated in slightly different ways that can, however, be classified within the description of the above mentioned procedures. Among these special mention is made to Refs. 57

and 58 which use respectively some known results in the theory of Bessel functions and a reformulation of the equations describing a parallel-flow heat exchanger process. By considering an observer riding a section of the fluid, with the second method it was possible to reduce significantly the amount of hardware required by the conventional finite differences analog simulation.

I. CONCLUSION

After having described so large a variety of techniques, the question arises: Which of them is the best?.

The answer is not a simple one. Depending on the available computers and also on the capabilities of each one (speed, size of the memory, number of components, accuracy, etc.), where a choice exists, the most convenient methods are those with which the user feels most familiar. It seems, however, that sufficiently large analog computers do not exist, to permit study of big or strongly nonlinear systems. In this case the only alternatives are the hybrid and the digital computers.

Table I synthesizes the main characteristics of the different forms of modeling just described.

TABLE I

COMPARISON OF THE DIFFERENT METHODS OF SIMULATING P.D.E.'S

Method	Computer	Recommended Type of P.D.E.'S	Comments
DSCT	Analog	Parabolic	If the system is strongly nonlinear there is a need for multiplexing.
CSCT	Hybrid or Analog	Flow and Hyperbolic	Unstable if forward time differences are used.
TSCT	Hybrid or Analog	Parabolic and Hyperbolic	Increasingly more popular (i.e. the Bubnov-Galerkin transformation).
TSCT	Hybrid or Digital	Any	If applied to wave equations with unknown initial conditions for the first derivatives, the use of an observer [Ref. 18, p. 72] becomes necessary
DSCT	Digital	Any	Very accurate. Applicable to very complex p.d.e.'s. The major drawback is the high cost in computer time.
DSTT	Hybrid or Digital	Unidimension Systems	Practical when the solution is desired at few points. An example is the infinite product expansion which preserves the max and min values of output.
CSTT	Hybrid or Analog	"	" " " " " "
TSTT	Any	_____	Difficult implementation.
MONTE-CARLO	Any	Any	The theory is out of the scope of this thesis. Requires small size memory or small analog computers.

III. OPTIMAL CONTROL OF UNIDIMENSIONAL DISTRIBUTED PARAMETER SYSTEMS

A. INTRODUCTION

In the sequel of the general lines formerly enunciated with this chapter it is intended to create a physical feeling for the optimal control of distributed parameter systems, by summarizing different types of approach to the problem; for simplicity only unidimensional systems will be considered. To accomplish this objective the solutions of five of the most representative problems are explained in depth.

As mentioned in Chapter I the present work will not be dealing with derivations of optimality conditions or with the mathematics of sensitivity, identification, stability, controllability and observability, although making superficial references to them. Each one of these topics has in itself the potential of an almost unlimited number of scientific dissertations.

According to a supposedly universal acceptance in this chapter the control function will always be represented by u and the optimal control by u^* . Similarly, all the optimal quantities (states and cost) will have a $*$ as superscript.

1. The Open-Loop Control

This has been the way most of the problems were worked. It requires the knowledge of the initial conditions and one specific field of application is in problems of known trajectory.

Schematically, the system representation is as follows, where the double lines represent distributed quantities, generally in vector form for one-dimensional systems and in matrix form for multidimensional ones. The optimal control is computed and stored in order to be used when necessary.

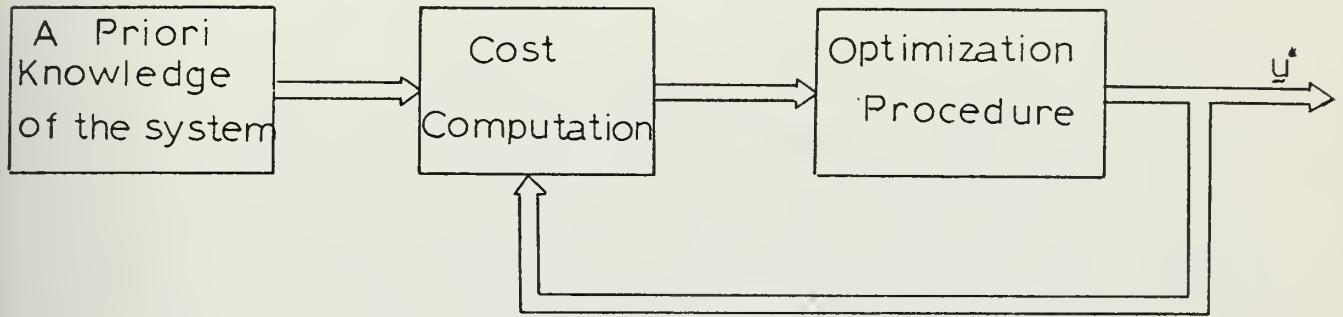


Figure 3.1. Open-Loop Control

Notice that now the control is a function of the states. With exception for the requirement on the knowledge of the initial conditions, all the statements in the last paragraph also apply here.

For linear systems with quadratic cost functions the optimal law is linear and an equation equivalent of the Ricatti equation for lumped parameter systems may be derived and then solved in closed form.

The cost is a well-defined function which can be computed in several ways, namely by numerical methods.

The only input to the system is the control action, which may be included in the boundary conditions or in the p.d.e. describing it. At this point any of the techniques described in Chapter II must be used.

The subject of the realization of the optimality conditions was left open. Not much of practical value has been said about such a topic; however, a large amount of theoretical developments involving Calculus of Variations and Functional Analysis is available in the literature.

Once again the approach may be divided; consider direct and indirect methods.

a. Direct Methods

The Direct Methods do not make use of the necessary conditions for optimality. They consider very often the change in the cost function (δJ) resultant from a small variation (δu) and, starting with a trial solution, they follow optimization techniques, such as the gradient method, until $\delta J > 0$ for $\delta u \leq 0$.

If the p.d.e.'s describing the system are reduced to ordinary differential or difference equation by techniques such as described in Chapter II (space quantization, time and space quantization, eigenfunction expansion, transfer function approximations, etc.), then the optimal control problem is reduced to a lumped parameter, one for which there are well known techniques.

b. Indirect Methods

The indirect methods may also use the known results of lumped parameters optimal control. However, either using these concepts, or deriving other results, they always require the knowledge of the necessary conditions for optimality. This will originate equations such as the Hamilton-Jacobi-Bellman, principles such as Pontriagyn's maximum

principle and operational procedures such as the moment method⁶ and dynamic programming.

Some other types of approach are possible, as for example the one shown by Balakrishnan [Ref. 6] which includes constraints in the form of a p.d.e. in the cost function.

2. The Closed-Loop Control

Its general configuration is shown in Fig. 3.2.

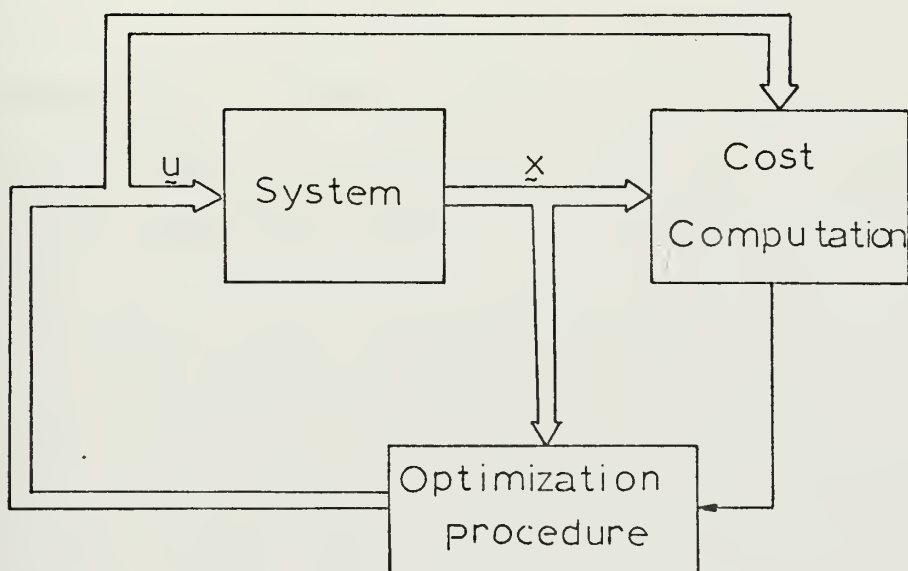


Figure 3.2. Feedback Control

B. PHYSICAL EXAMPLES OF OPTIMAL CONTROL OF D.P.S.'s

1. Heating Problems

Suppose a large heating furnace used to heat metal billets is given and that these billets move uniformly through

⁶ This method was formulated in general terms by Kreyn in 1938 and used by Butkovskiy. It is a good method within its scope of application: linear systems with known eigenfunctions.

the processing zone. The requirement imposed on the plant, besides the cost, is that the outlet material must be at constant temperature. In order to be able to implement a good control system the engineer has to take in account the time a given section of the material takes to cross the furnace, the temperature distribution, the physical properties of the billets, as well as its thickness, etc. Intuitively, the possible control laws would take the form of changing the speed, or the temperature, or both.

There are numerous constraints to be taken in consideration, as for instance in the furnace temperature $T_1(x,t)$, such that $C_2 \leq T_1(x,t) \leq C_1$, where C_1 and C_2 are constants and x is the distance from origin. Also the interaction between adjacent zones of the furnace must obey the inequality

$\frac{\partial T_1(x,t)}{\partial x} \leq C_3$, the billets temperature $T_2(x,y,t)$ must be less than a constant (here y is measured in the direction of the thickness) and, similarly, $\frac{\partial T_2(x,t)}{\partial x} \leq C_4$.

2. Heat and Mass Transfer Problems

One example of this type of problems is the process of drying of a moist material in a continuous drying unit. In this case the control action consists in compensating the changes in the composition of the moisture, porosity of the material, layer thickness, velocity of the material inside the heater, etc. The constraints may assume different aspects, a typical example being $m \leq u(x,t) \leq M$, $|\frac{\partial u}{\partial x}| \leq C_1$ and the temperature inside the material also less than a constant.

3. Separation of Liquids and Gaseous Mixtures

When using big reactors their distributed nature must be considered and once again the D.P.S.'s techniques are required. The problem may be formulated in three different ways, in terms of the control action:

- (i) Control of the source heat to obtain the highest concentration of the substance, assuming a constant output rate;
- (ii) Control of the source heat to obtain a maximum product output for a given concentration of the mixture; and
- (iii) Minimization of the control costs for a given concentration and output rate of the product.

4. Automatic Control of Large Hydroturbines

In this case, the distributed parameter nature of the pipeline supplying water to the turbine has to be considered. The control of the turbine rotor is realized with a valve, but while this action should be fast, in order to minimize abrupt changes in velocity when the load is removed, it has to be limited by the resulting changes in pressure on the pipeline walls, which cannot exceed the safety value.

5. Gas Transfer Through Long Pipelines

The problem consists in locating and controlling compressors along the pipeline such that the variations in the output pressure are minimized. Considering only one compressor situated at the origin (single pointwise control) the boundary conditions will be of the type

$$\begin{aligned}p(0,t) &= u(t) \\v(l,t) &= v_a(t) ,\end{aligned}$$

where $p(0,t)$ is the pressure at $x=0$ and $v(l,t)$ is the gas velocity at $x=l$. This velocity equals $v_a(t)$ which in turn is a function of the load.

The functional to be extremized is of the type

$$J = \int_0^T |p^*(t) - p(l,t)|^\gamma dt, \quad \gamma \geq 1$$

The control of systems such as those just described is by itself a challenging task due to the complexity of the equations involved. Unfortunately, this task is made even more difficult by the fact that some of the parameters of the partial differential equation are not accessible, which implies the need for the use of parameter identification techniques.

C. FIVE REPRESENTATIVE EXAMPLES

A brief analysis of the contents of this section is given:

Example No. 1 [Ref. 5] is an open-loop indirect method applicable to a large variety of D.P.S.'s for which a pointwise control is desired. It is based on a space discretization technique that leads to a set of ordinary differential equations to which the methods of optimal control of lumped parameter systems can be applied.

Example No. 2 [Ref. 5] has similar characteristics to the preceding example but is a multiple pointwise control problem.

Example No. 3 [Ref. 56] is the last of the open-loop problems presented here. It is also an indirect method and may be used whenever the partial differential equation may be reduced to

the form $Q(x,s)=G(x,s)U(s)$, where $Q(x,s)$ and $U(s)$ are respectively the transforms of the output and of the input.

Example No. 4 [Ref. 29] is one of the few closed-loop methods available in the literature. It applies to parabolic differential equations with quadratic cost-function. The control is multiple pointwise.

Example No. 5 [Ref. 28] presents again a closed-loop technique. It applies to partial differential equations with known analytical solution and when the functional cost is quadratic. In the example shown there is only a single control but a multiple pointwise control is also possible.

1. Open-Loop Minimum Time Optimal Control of a Heat Transfer System Using Space Discretization and Pontryagin's Maximum Principle

Consider the heat equation

$$\frac{\partial q}{\partial t} = k \frac{\partial^2 q}{\partial x^2}, \quad 0 \leq x \leq L \text{ and } 0 \leq t \leq T \quad (3.1)$$

The boundary conditions are

$$-a \frac{\partial q}{\partial x} \Big|_{x=0} = b[u(t) - q(0,t)] \quad (3.2)$$

where a and b are given constants, and

$$\frac{\partial q}{\partial x} \Big|_{x=L} = 0 \quad (3.3)$$

The initial condition is

$$q(x,0) = q_{I.C.}(x) \quad (3.4)$$

The pointwise control is bounded: $|u(t)| \leq M$

One way of reformulating a minimum time problem is to choose a convenient value for the final (t_f) such that at this instant the deviation from the optimal trajectory is minimum. An adequate cost function is given by

$$J = \int_0^L |q^*(x) - q(x,t)|^\gamma dx, \quad \gamma \geq 1 \quad (3.5)$$

Discretizing equations 3.1 through 3.4, the following expressions result for $L=N\ell$:

$$\frac{dq_1}{dt} = k \frac{\frac{q_2 - q_1}{\ell} - \frac{q_1 - q_0}{\ell/2}}{\ell}$$

$$\frac{dq_i}{dt} = k \frac{\frac{q_{i+1} - q_i}{\ell} - \frac{q_i - q_{i-1}}{\ell}}{\ell}, \quad i=2,3,\dots,N-1 \quad (3.1a)$$

$$\frac{dq_n}{dt} = k \frac{\frac{q_{N+1} - q_N}{\ell/2} - \frac{q_N - q_{N-1}}{\ell}}{\ell}$$

$$- a \frac{q_1 - q_0}{\ell/2} = b(u - q_0) \quad (3.2a)$$

$$\frac{q_{N-1} - q_N}{\ell/2} = 0 \quad (3.3a)$$

This is clearly illustrated in Fig. 3.3.

The above discrete equations can be written in dimensionless form:

$$q_0 = \frac{\beta}{2+\beta} u + \frac{2}{2+\beta} q_1, \quad \text{for } \beta = \frac{b\ell}{a} \quad (3.2b)$$

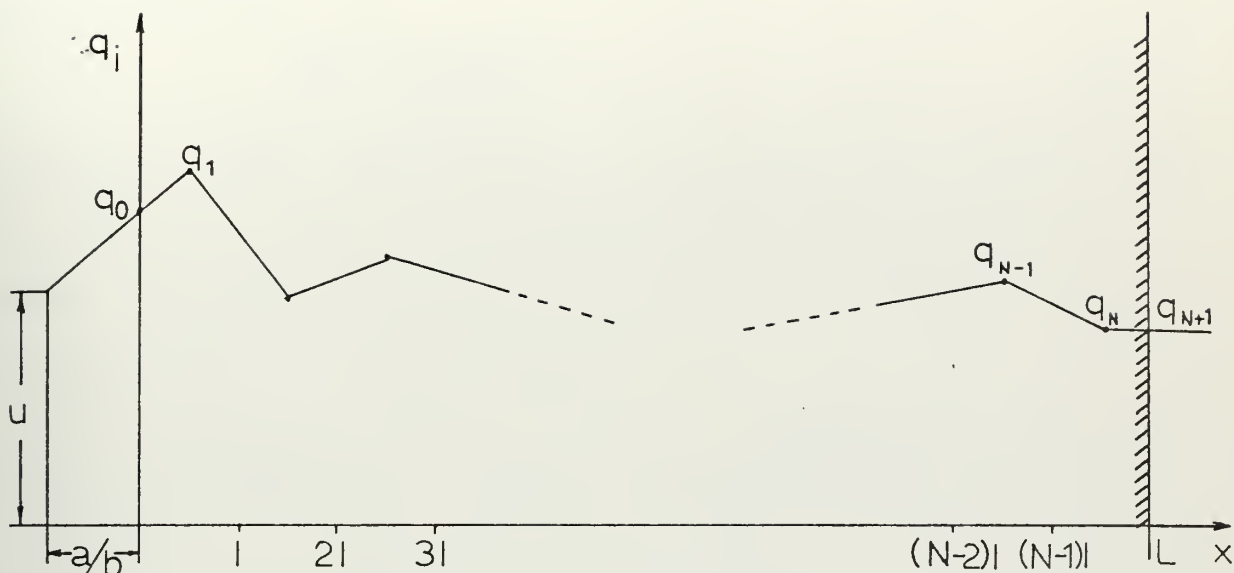


Figure 3.3. Method of the Straight Lines

$$\frac{dq_1}{d\tau} = 2q_0 - 3q_1 + q_2, \quad \tau = \frac{2t}{\ell^2}$$

$$\frac{dq_i}{d\tau} = q_{i-1} - 2q_i + q_{i+1}, \quad i=2,3,\dots,N-1 \quad (3.1b)$$

$$\frac{dq_N}{d\tau} = q_{N-1} - q_N$$

Substituting (3.2b) in (3.1b) the result is

$$\frac{dq_1}{d\tau} = \frac{2\beta}{2+\beta}u - \frac{2+3\beta}{2+\beta}q_1 + q_2$$

$$\frac{dq_i}{d\tau} = q_{i-1} - 2q_i + q_{i+1}, \quad i=2,3,\dots,N-1 \quad (3.1c)$$

$$\frac{dq_N}{d\tau} = q_{N-1} - q_N$$

which is easily implemented on an analog computer.

The given p.d.e. may also be simulated easily remembering that

$$\frac{\partial q}{\partial \tau} = \frac{1}{rc} \frac{\partial^2 q}{\partial x^2} \quad (3.6)$$

gives the voltage distribution in a non-inductive transmission line that has r and c as resistance and capacitance per unit length. Therefore the total resistance and capacitance are $R=rL$ and $C=cL$.

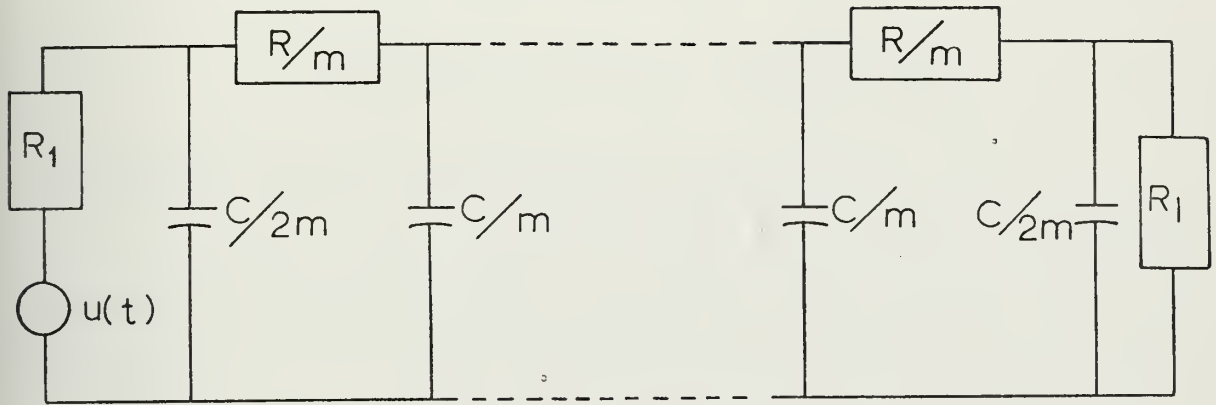


Figure 3.4. An Analogic Simulation of a p.d.e.

The equivalent m sections circuit is represented in Fig. 3.4 where $\frac{R}{R_1} = 2\beta N$, $RC = \frac{L^2}{a}$ (3.7) and the load $R_\ell = \infty$ because of Eq. 3.3.

Returning to Eq. 3.1c, suppose that the following initial conditions are given

$$q_0(0) \quad (3.8)$$

$$q_i(0) = q_0(0) \left(\frac{2i-1}{2} \ell \right), \quad i=1,2,\dots,N$$

Equation 3.5 reformulated for the discrete case becomes

$$J = \sum_{i=1}^n q_i^2(t_f) \quad (3.9)$$

which is to be maximized.

The system (3.1c) is representable in the linear form as

$$\dot{\underline{x}}(t) = \underline{A}(t) \underline{x}(t) + \underline{B}u \quad (3.10)$$

and, as a consequence, the necessary condition for optimality from Pontryagin's principle becomes also the sufficient condition.⁷

Writing the Hamiltonian⁸

$$\begin{aligned}
 H = & -1 + p_1 \left(\frac{2\beta}{2+\beta} u - \frac{2+3\beta}{2+\beta} q_1 \right) + q_2 \\
 & + \sum_{i=2}^{n-1} p_i (q_{i-1} - 2q_i + q_{i+1}) + p_N (q_{N-1} - q_N)
 \end{aligned} \quad (3.11)$$

where the p's are the Lagrange multipliers.

As it is easily seen, given that $|u| \leq M$, the Hamiltonian has a maximum for

$$u(t) = M \operatorname{sign} p_1(t) \quad (3.12)$$

Looking now at \underline{A} of Eq. 3.10:

$$\underline{A} = \begin{pmatrix}
 -\frac{2+3\beta}{2+\beta} & 1 & 0 & 0 & \text{----} & 0 & 0 \\
 1 & -2 & 1 & 0 & \text{----} & 0 & 0 \\
 \text{-----} & & & & & & \\
 0 & 0 & 0 & 0 & -2 & 1 \\
 0 & 0 & 0 & 0 & 1 & -1
 \end{pmatrix} \quad (3.13)$$

It should be noticed that it is symmetric and therefore its eigenvalues are real. It also can be shown that the eigenvalues are all negative. This last result shows that the

⁷ Reference 65, p. 234.

⁸ Reference 37, p. 188.

system moves to the origin as $\tau \rightarrow \infty$ and this implies the existence of a unique optimal control. Because the eigenvalues are real and negative the optimal control will change its value n times, as illustrated in Fig. 3.5 for $n=4$. The initial and final conditions are $q_i(0)=-1$ and $q_i(T)=0$ for $i=1,2,3,4$,

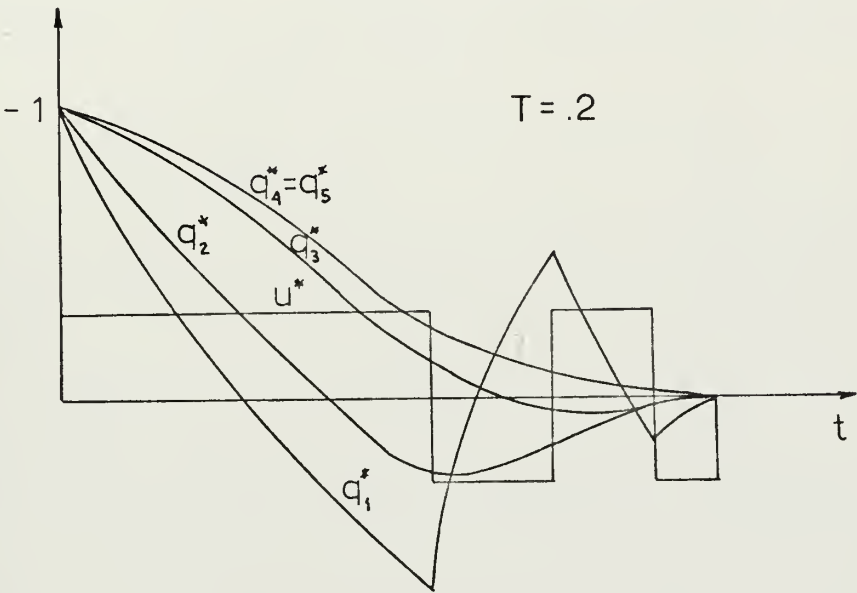


Figure 3.5. The Optimal Control Law and the Correspondent Output.

2. A Simple Example with Open-Loop Multiple Pointwise Control

Consider the heat transfer equation

$$a(x,t) \frac{\partial q}{\partial t} + a(x,t)v(t) \frac{\partial q}{\partial t} + q = u(x,t) \tag{3.14}$$

between a medium at temperature u and a body moving with velocity $v \geq 0$. The heat takes place between the x origin and $x=L$.

The remaining information is: a and v are known and

$$q(x,0) = q_{I.C.}(x) \tag{3.15}$$

$$q(0,t) = 0 \quad , \quad 0 \leq t \leq T \tag{3.16}$$

The problem consists in finding the control $u(x,t)$ such that the functional

$$J = \int_0^T |q^*(t) - q(L,t)|^\gamma dt, \quad \gamma \geq 1 \quad (3.17)$$

is minimized or equivalently, such that the time integral of the temperature deviation of the output material from a given law is minimized. The following constraints are imposed:

$$\begin{aligned} C_2 &\leq u(x,t) \leq C_1 \\ C_4 &\leq \frac{\partial u}{\partial x} \leq C_3 \end{aligned} \quad (3.18)$$

Discretizing the system in n linear elements, Eq.

3.14 becomes

$$a_i(t) \frac{dq_i}{dt} + a_i(t)v(t) \frac{q_i - q_{i-1}}{\ell} + q_i = u_i(t) \quad (3.19)$$

But, by Eq. 3.16, $q_0(t) = 0$ and therefore it follows

$$\frac{dq_i}{dt} = \beta q_{i-1} + \alpha_i q_i + \frac{u_i}{a_i}, \quad i=1,2,\dots,N \quad (3.20)$$

where

$$\beta = \frac{v(t)}{\ell} \text{ and } \alpha_i = -\frac{1}{a_i(t)} - \frac{v(t)}{\ell} \quad (3.21)$$

In order to have the term with u_i free of other time functions define $q'_i = a_i q_i$ and rewrite Eq. 3.21 as

$$\frac{dq'_i}{dt} = \beta q'_{i-1} + \alpha_i q'_i + u_i \quad (3.22)$$

The discrete form of the constraints becomes

$$\begin{aligned} C_2 &\leq u_i(t) \leq C_1 \\ C_4 &\leq u_{i+1} - u_i \leq C_3 \end{aligned} \quad (3.23)$$

and the cost takes the form

$$J = \int_0^T |q'^* - q'_n(t)|^\gamma dt, \quad \gamma \geq 1 \quad (3.24)$$

The corresponding Hamiltonian is

$$H(p, q, u) = p_0 |q'^*(t) - q'_n(t)|^\gamma + \sum_{i=1}^N p_i (\beta q'_{i-1} + \alpha_i q'_i + u_i) \quad (3.25)$$

and, as usual, the costate equations are obtained by

$$\frac{dp(t)}{dt} = - \frac{\partial H}{\partial q'}$$

or

$$\frac{dp_0^*}{dt} = 0$$

$$\frac{dp_i^*}{dt} = -\alpha_i p_i - \beta p_{i+1}, \quad i=1, 2, \dots, N \quad (3.26)$$

$$\frac{dp_N^*}{dt} = -\gamma p_0 |q'^* - q'_N|^{\gamma-1} \text{sign}(q'^* - q'_N) - \alpha_N p_N$$

As it may be immediately seen, for $p_i = \text{const.}$ H takes the maximum value for a corresponding maximum of

$$F(p_i, u_i) = \sum_{i=1}^N p_i^* u_i \quad \text{satisfying the constraints in Eq. 3.23.}$$

If $L=2\ell$ it is possible to draw one elucidative two-dimensional picture showing the effect of the constraints on the optimal control. Equations 3.26 together with

$$\frac{\partial q^*}{\partial t} = \frac{\partial H}{\partial p}, \quad \text{the control constraints and } H(q^*(T), \underline{u}^*(T), \underline{p}^*(T), T) \delta T = 0^9 \quad \text{lead to the complete solution of the problem.}$$

⁹ Reference 37, p. 233.

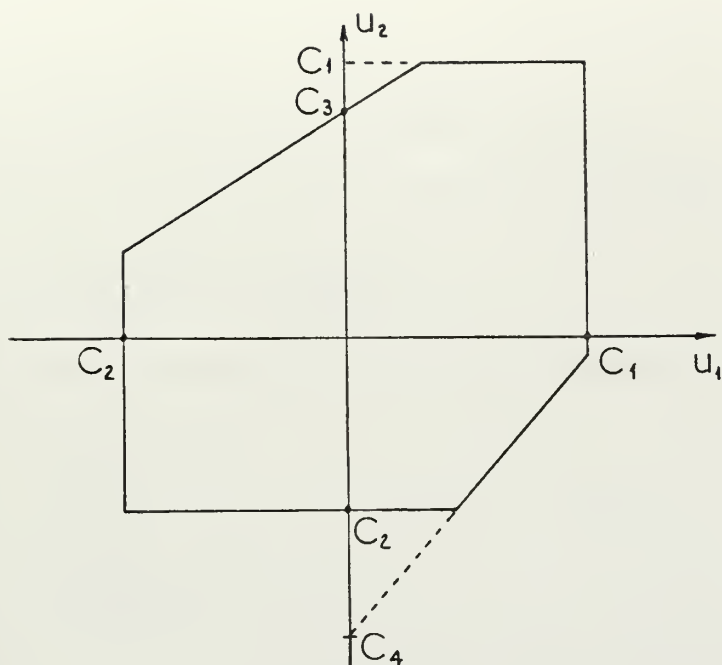


Figure 3.6. Geometrical Interpretation of the Control Constraints.

Butkovskiy [Ref. 5] describes two other methods used to find the switching times of the control in minimum time problems. They are the method of harmonics and the method of parabolic approximations. Both give better accuracy than the straight lines approximation illustrated in Fig. 3.3 and they are mentioned here as a reference to the interested reader.

Finally, since there are few references to multi-dimensional systems, it seems useful to refer to the above mentioned work for an explanation of a simple way of reducing the problem of optimal heat of a sphere or of a cylinder to the one-dimensional problem using the method of harmonics.

3. Sakawa's Solution of One Open-Loop Optimal Heat Transfer Problem

Sakawa's solution [Ref. 56] is one of the most comprehensive and complete papers in the field of practical applications of distributed parameter systems.

Given the normalized parabolic equation

$$\frac{\partial^2 q(x,t)}{\partial x^2} = \frac{\partial q(x,t)}{\partial t} , \quad (3.27)$$

for $0 \leq x \leq 1$ and $0 \leq t \leq T$ with initial and boundary conditions

$$q(x,0) = 0$$

$$\frac{\partial q(x,t)}{\partial x} \Big|_{x=0} = \alpha \{q(0,t) - v(t)\} , \quad (3.28)$$

$$\frac{\partial q(x,t)}{\partial x} \Big|_{x=1} = 0$$

the control $u(t)$, $0 < t < T$, is a fuel flow which controls $v(t)$, the temperature of the medium.

Equation

$$\gamma \frac{dv(t)}{dt} + v(t) = u(t) \quad (3.29)$$

expresses the relation between $u(t)$ and $v(t)$ and is physically equivalent to a first-order lag.

The objective is to minimize the functional

$$J(u(t)) = \int_0^1 \{q^*(x) - q(x,T)\}^2 dx \quad (3.30)$$

where $q^*(x)$ represents the desired temperature distribution at $t=T$.

Applying Laplace transform techniques to Eqs. 3.27 and 3.29 and solving for the given boundary conditions, $Q(x,s)$ may be written as $Q(x,s) = G(x,s)U(s)$.

From this equation, by the convolution theorem

$$\begin{aligned} q(x,t) &= \int_0^t g(x,\tau) u(t-\tau) d\tau \\ &= \int_0^t g(x,t-\tau) u(\tau) d\tau \end{aligned} \quad (3.31)$$

Note: All the procedures that follow may be applied not only to the heat equations, but to any partial differential equation reducible to the form of Eq. 3.31.

Rewriting $G(x,s)$ as the quotient $G(x,s) = \frac{N(x,s)}{N(s)}$

it is easy to use the theorem of the residues and obtain

$$g(x,t) = \sum_{i=0}^{\infty} \frac{N(x,s_i)}{M'(s_i)} e^{s_i t} \quad (3.32)$$

where the $M'(s_i)$ is the derivative of $M(s)$ at $s=s_i$

By the use of standard minimization techniques described in detail in the reference paper, from Eq. 3.30, the optimality condition can be derived:

$$\int_0^1 \{q^*(x) - q(x,t)\} g(x,T-\tau) dx = 0 \quad (3.33)$$

$$\text{Defining } f(\tau) = \int_0^1 q^*(x) g(x,T-\tau) dx \quad (3.34)$$

it comes out, by bottom Eqs. 3.31 and 3.33:

$$f(\tau) = \int_0^T u(\mu) \int_0^1 g(x,T-\tau) g(x,T-\mu) dx d\mu \quad (0 \leq \tau \leq T) \quad (3.35)$$

where μ is a time dummy variable.

The integral on the right

$$y(\tau,\mu) = \int_0^1 g(x,T-\tau) g(x,T-\mu) dx d\mu \quad (3.36)$$

is a symmetric kernel¹⁰ and as such has some important properties:

(i) There exists at least one eigenvalue, $\lambda_i \neq 0$ ¹¹

$\phi_i(\tau) = \lambda_i \int_0^T y(\tau, \mu) \phi_i(\mu) d\mu$ where $\phi_i(\tau)$ ($0 \leq \tau \leq T$) is the eigenfunction corresponding to λ_i .

(ii) The eigenfunctions are mutually orthogonal, i.e.,

$$\int_0^T \phi_i(\tau) \phi_j(\tau) d\tau = 0, \quad \text{for } \lambda_i \neq \lambda_j.$$

(iii) The necessary and sufficient conditions for existence

of solutions of Eq. 3.35 is that $\sum_{i=1}^{\infty} \lambda_i^2 C_i^2$ must be convergent, where $C_i = \int_0^T f(\tau) \phi_i(\tau) d\tau$. The optimal control will be given by:

$$u^*(\tau) = \sum_{i=1}^{\infty} \lambda_i C_i \phi_i(\tau). \quad (3.37)$$

From Eq. 3.37 it can be seen that in order to obtain the optimal control it is necessary to solve also Eq. 3.35 which, in general, is a hard task. Also it happens that sometimes the solution of Eq. 3.35 does not account for control restrictions. For these reasons Sakawa develops a numerical integration of Eqs. 3.31 and 3.32 obtaining

$$J(u) \approx \sum_{i=0}^N C_i (q_i^* - \sum_{j=0}^n a_{ij} u_j)^2 \quad (3.38)$$

¹⁰ See Ref. 11, p. 115, for detailed definition and properties.

¹¹ \supset means "such that."

where the C_i 's and a_{ij} 's are obtained in a straightforward manner. A way of minimizing Eq. 3.38 is by quadratic programming, for which Sakawa gives appropriate references. However, because a linear programming technique is much easier to implement, he considers another cost function:

$$J(u(t)) = \int_0^1 |q^*(x) - q(x,t)| dx \quad (3.39)$$

From this he derives, in a similar fashion as for Eq. 3.38, the following equation:

$$J(u) \approx \sum_{i=0}^n C_i |q_i^* - \sum_{j=0}^N a_{ij} u_j| \quad (3.40)$$

The control constraints are taken in account in the linear (or nonlinear) programming.

In the sequence of this last procedure the author obtained the following curves for the parameters

$$\alpha = 10, \quad \gamma = 0.04, \quad q^*(x) = .2, \quad n = 20$$

Summarizing, Sakawa's method is considered to be quite valuable for single pointwise control of unidimensional systems. Although it is not impossible to generalize this method for multiple control using multiple Laplace transforms and the superposition principle, this seems to be a huge task and therefore the method is not recommended for such cases.

Quite recently, Chang [Ref. 9, August 1970] developed one algorithm based on a modified steepest descent method and solves the same problem as above obtaining very good agreement with only two iterations.

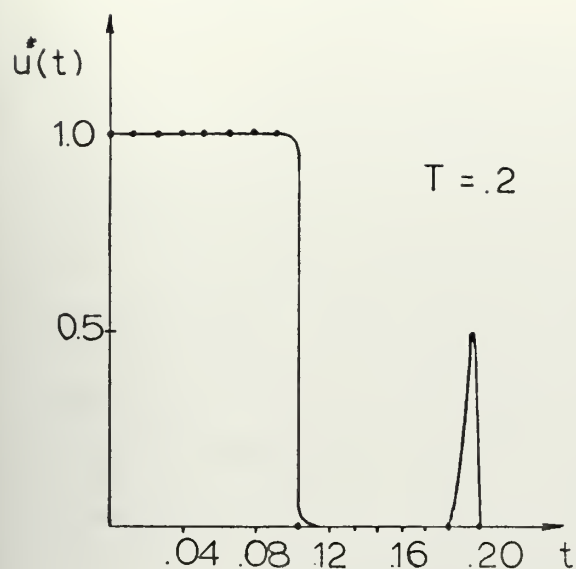


Figure 3.7. Optimal Control
($T = .2$).

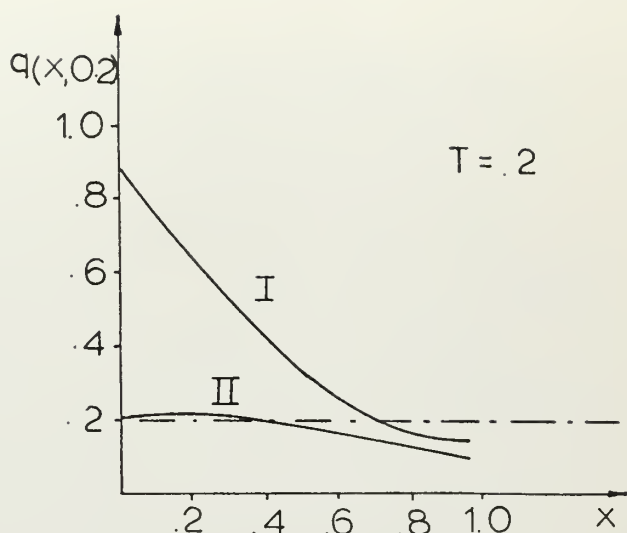


Figure 3.8. I. Temperature
Distribution for $u(t) = 1$.
II. Temperature Distribution
for $u^*(t)$.

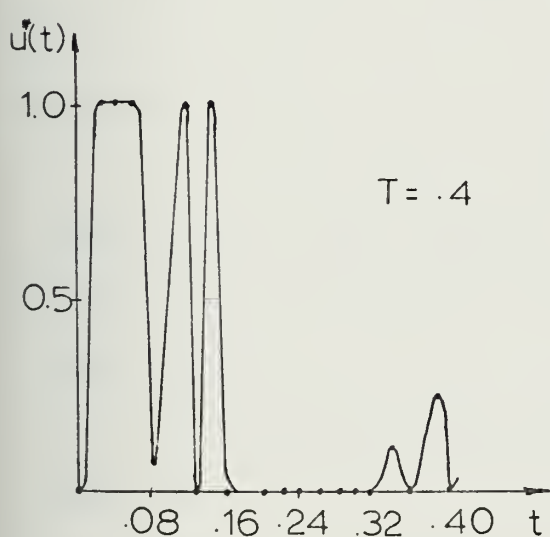


Figure 3.9. Optimal Control
($T = .4$).

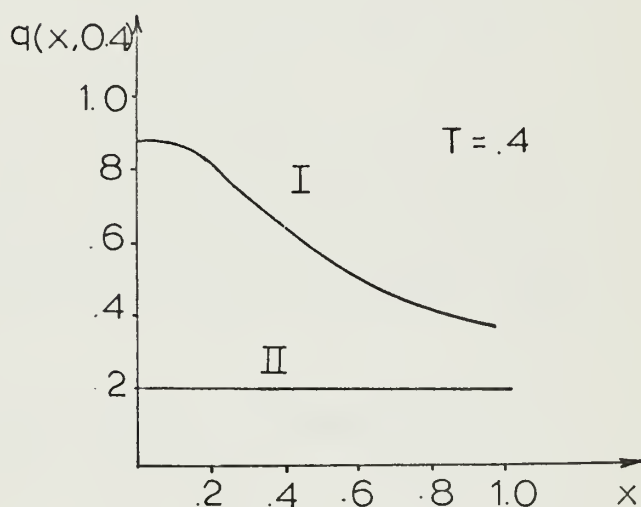


Figure 3.10. I. Temperature
Distribution for $u(t) = 1$.
II. Temperature Distribution
for $u^*(t)$.

4. Greenberg's Solution of a Closed-Loop Multiple Pointwise Optimal Control Problem with Quadratic Cost Function

a. Pointwise Optimal Control

The mathematical derivations used by Greenberg in his paper [Ref.29] are hard to follow if the reader does not have a deep understanding of Functional Analysis and theory of operators. However, the consequences of this method are so important and its implementation is so simple, if compared with the complexity of the derivations, that it is worthwhile to analyze its contents in some detail.

The typical approach to the problem represents a distributed control law by a finite number of lumped controls, but this is not applicable to a feedback control. Also the modal control, by the reasons already pointed out in Chapter I, is difficult to realize if the eigenvalues are hard to obtain or if they converge slowly. As it will be shown, the present method does not suffer such drawbacks.

The author starts with the knowledge previously demonstrated that systems described by parabolic differential equations with quadratic cost functions have an optimal feedback control.

Writing the parabolic differential equation in the form:

$$\dot{q}(t) = Aq(t) + B(t)u(t) , \quad q(0) = q_0 , \quad (3.41)$$

where A is a partial differential operator (generally of the type $\frac{\partial^2}{\partial x^2}$) and $B(t)$ is a time variant control operator defined for $D \leq t \leq T$, the functional cost is defined as

$$J = \int_0^T [\langle Q(t)q(t), q(t) \rangle + \tilde{u}^T(t) \tilde{R}(t) \tilde{u}(t)] dt \quad (3.42)$$

$\tilde{R}(t)$ is a $k \times k$ positive definite matrix and $Q(t)$ a positive operator; k represents the number of desired control points.

After non-trivial manipulations the optimal control law is found:

$$\tilde{u}^*(t) = - \tilde{R}^{-1}(t) \tilde{B}(t) \int_D \hat{k}(t, \zeta) q(t, \zeta) d\zeta \quad (3.43)$$

where $\hat{k}_i(t, x) = K(t, x, x_i)$, $i = 1, 2, \dots, k$ with $K(t, x, \zeta)$ the solution of a given integro - differential equation, and

$\tilde{B}(t)$ a $k \times k$ diagonal matrix, with easily computable elements.

The block diagram correspondent to the above mentioned description is represented in Fig. 3.11.

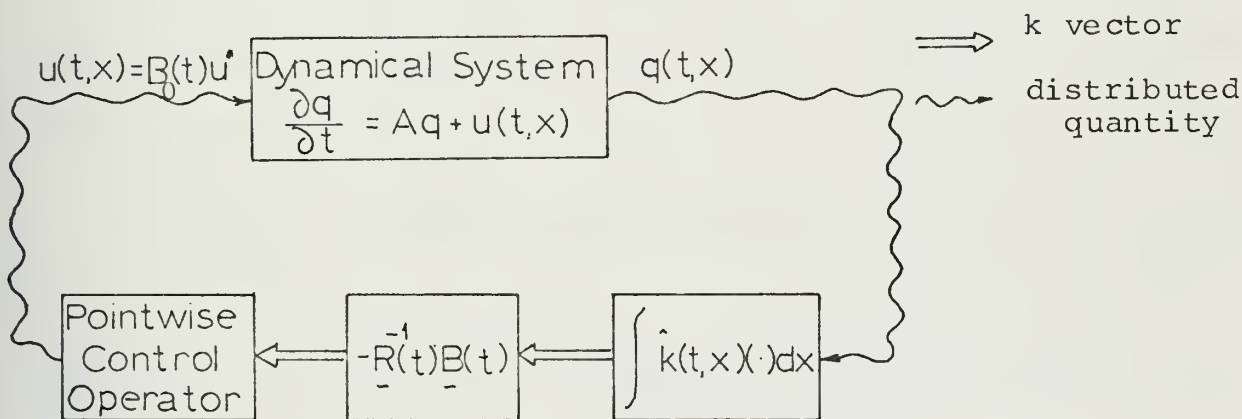


Figure 3.11. Block Diagram for Greenberg's Method

¹² The symbol $\langle x, y \rangle$ means inner product of the vectors \tilde{x} and \tilde{y} .

$B_0(t)$ is defined using spectral decomposition of operators, a subject that will be treated in detail in the next chapter.

b. Infinite Terminal Time Problem

The above description can be quite simplified in the case of a terminal time problem because the operators B_0 , Q , R and K the feedback operator become time invariant.

Represent

$$Q(x, \zeta) = \tilde{V}^T(x) \tilde{Q} \tilde{V}(\zeta) \quad (3.44)$$

where $V(x)$ is the vector of the first \underline{N} eigenfunction of A and \tilde{Q} a $N \times N$ positive definite matrix. Define \tilde{K} as the positive definite solution of the matrix Riccati equation $-\tilde{\Lambda} \tilde{K} - \tilde{K} \tilde{\Lambda} + \tilde{K} \tilde{V}^T \tilde{B} \tilde{R}^{-1} \tilde{B} \tilde{V} \tilde{K} - \tilde{Q} = 0$, with $\tilde{\Lambda}$ a $N \times N$ diagonal matrix such that Λ_{ii} equals λ_i of A . Also define V as a $k \times N$ matrix whose ij^{th} element is $V_{ij} = v_j(x_i)$. The optimal control is given by

$$\begin{aligned} u^*(t) &= - \tilde{R}^{-1} \tilde{B} \tilde{V} \tilde{K} \int_D \tilde{V}(\zeta) q(t, \zeta) d\zeta \\ &= \tilde{R}^{-1} \tilde{B} \tilde{V} \tilde{K} \tilde{q}(t) \end{aligned} \quad (3.45)$$

where $\tilde{q}(t)$ is the N vector of modal coefficients of the state distribution $q(t, x)$.

Due to Eq. 3.44, if the optimal control exists it cannot be improved by feeding back more than \underline{N} modes. Another important conclusion, is that whenever $(Q_{N+1} - Q_N)$ is a positive operator it follows that $K_{N+1} - K_N \geq 0$. This can be stated as "monotone approximation of the state weighting operator originates monotone approximation of the optimal feedback operator."

Greenberg finishes his paper with a very simple and illustrative example.

5. Graham's Solution of a Close-Loop Pointwise Optimal Control Problem

J. W. Graham's paper [Ref. 28] presents two different approaches to the solution of an optimal control problem characterized by a given parabolic differential equation with a frequent type of initial and boundary conditions. The author uses two types of approach to the problem; the first one deals with the matrix Ricatti's equation and the second one considers Kalman's equation, together with root-locus techniques, constituting, in a certain way, like a smooth transition between the classical and the optimal control problems.

Some details in the derivations are very hard to follow due to gaps in the explanatory theory; for this reason additional information will be given in an attempt to clarify the points where more omissions were found. The nomenclature used is consistent with the remaining of this chapter, but the reverse of the one in the paper regarding the control and the temperature.

The p.d.e.

$$\frac{\partial q}{\partial t}(x,t) = v^2 \frac{\partial^2 q}{\partial x^2}, \quad t > 0, \quad 0 < x < a \quad (3.46)$$

is given with initial conditions

$$q(x,0) = Q_0(x) \quad (3.47)$$

and boundary conditions (B.C.'s)

$$\begin{aligned} \frac{\partial q}{\partial x}(0,t) &= 0 \\ K \frac{\partial q}{\partial x}(a,t) &= f(t) - q(a,t) = u(t) \end{aligned} \quad (3.48)$$

where K is the thermal conductivity and $f(t)$ is an external available temperature input. All the other symbols are defined as before.

The cost function is defined as

$$J = \int_0^{\infty} [q(a,t) - q_{ss}(a)]^2 + R[u(t) - u_{ss}]^2 dt \quad (3.49)$$

where the subscript ss represents steady-state and R is a positive constant. The physical meaning of the equation is obvious.

The analytic solution of the p.d.e. for the given boundary conditions and piecewise constant control ($u(kt)$) is given (although not necessary):

$$\frac{q(x,t)}{u(kT)} = Ka \left\{ \frac{v^2 t}{a} + \frac{3x^2 - a^2}{6a^2} - \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \exp\left[-\frac{v^2 n^2 \pi^2 t}{a^2}\right] \cos \frac{n\pi x}{a} \right\} \quad (3.50)$$

for $[kT \leq t < (k+1)T]$.

After this the author states: "The original sampled-data equations in discrete form are transformed to the normal or diagonal form in a straightforward manner and from these equations a continuous-time set of equations is readily obtained. Hence the system equations may be written as the following:

$$\tilde{W}(t) = \tilde{\Lambda} \tilde{W}(t) + \tilde{B} u(t) \quad (3.51)$$

$$Q(t) = \tilde{C} \tilde{W}(t) \quad (3.52)$$

where $\tilde{W}(t)$ is an N dimensional column vector, $\tilde{\Lambda}$ a $N \times N$ diagonal matrix, \tilde{B} and N -dimensional column vector, $Q(t)$ the temperature of the system at $x=a$ and \tilde{C} is an N -dimensional vector."

The way Eqs. 3.51 and 3.52 were obtained does not seem so obvious as the author pretends. One possible technique that can be used to find them is the Bubnov-Galerkin method described in Appendix A.

Back to the original problem, the functional J can be further simplified by defining

$$\begin{aligned} Q^*(t) &= Q(t) - Q_{ss} \\ u^*(t) &= u(t) - u_{ss} \end{aligned} \quad (3.53)$$

from which

$$\begin{aligned} \dot{\tilde{W}}^*(t) &= \tilde{A} \tilde{W}^*(t) + \tilde{B} u^*(t) \\ Q(t) &= \tilde{C} \tilde{W}^*(t) \end{aligned} \quad (3.54)$$

with initial conditions

$$\begin{aligned} Q^*(0) &= -Q_{ss} \\ W^*(0) &= -W_{ss} \end{aligned} \quad (3.55)$$

and it comes out

$$J = \int_0^{\infty} [Q^*(t)^2 + R u^*(t)^2] dt \quad (3.56)$$

or

$$J = \int_0^{\infty} [\tilde{W}^*(t)^T \tilde{S} \tilde{W}^*(t) + R u^*(t)^2] dt \quad (3.57)$$

where

$$\tilde{S} = \begin{bmatrix} c_1^2 & c_1 c_2 & \dots & c_1 c_N \\ c_2 c_1 & c_2^2 & \dots & c_2 c_N \\ \vdots & \vdots & \ddots & \vdots \\ c_N c_1 & c_N c_2 & \dots & c_N^2 \end{bmatrix} \quad (3.58)$$

and the c_i 's ($i=1,2,\dots,N$) are the elements of the C vector. Note that in the above equations the superscript (*) is a mean for identification of some variables after

transformations on them, rather than being the usual convention for optimal function.

At this point the author considers two different approaches to the computation of the optimal control.

a. Matrix Riccati Equation Method of Solution

Because of the linearity of the given system and also because a quadratic cost function was chosen, it is known that the computation for the optimal control may be reduced to the solution for \tilde{P} of the Riccati equation [Ref. 3, p. 752-775]

$$\dot{\tilde{P}}(t) + \tilde{S} - \frac{\tilde{P}(t)\tilde{B}\tilde{B}^T\tilde{P}(t)}{\tilde{R}} + \tilde{P}(t)\tilde{A} + \tilde{A}^T\tilde{P}(t) = 0 \quad (3.59)$$

with boundary condition $\tilde{P}(t_f) = 0$. This boundary condition comes from the definition of J . If J had another quadratic term outside the integral, i.e. the term $\tilde{W}^{*T}\tilde{G}\tilde{W}^*$, the boundary condition would be $\tilde{P}(t_f) = \tilde{G}$.

The optimal control law is given by

$$u^*(t)_{\text{opt}} = -\tilde{R}^{-1}\tilde{B}^T\hat{\tilde{P}}\tilde{W}(t) \quad (3.60)$$

where

$$\hat{\tilde{P}} = \lim_{t_f \rightarrow \infty} \tilde{P}(t) \quad (3.61)$$

whenever the system is controllable and has $\tilde{P}(t_f) = 0$.

$\hat{\tilde{P}}$ is obtained as the solution of

$$-\hat{\tilde{P}}\tilde{A} - \tilde{A}^T\hat{\tilde{P}} + \hat{\tilde{P}}\tilde{B}\tilde{R}^{-1}\tilde{B}^T\hat{\tilde{P}} - \tilde{S} = 0, \quad (3.62)$$

or by solving Eq. 3.59 backwards in time from the known condition $\tilde{P}(\infty) = 0$ until convergence is reached. This last

method can be implemented very fast in a digital computer using the algorithm

$$\tilde{P}(t+\Delta) \approx \tilde{P}(t) + \Delta \{ -\tilde{P}(t)\tilde{A} - \tilde{A}\tilde{P}(t) + \tilde{P}(t)\tilde{B}\tilde{R}^{-1}\tilde{B}^T\tilde{P}(t) - \tilde{S} \} \quad (3.63)$$

The graphical interpretation of the procedure is given in Fig. 3.12

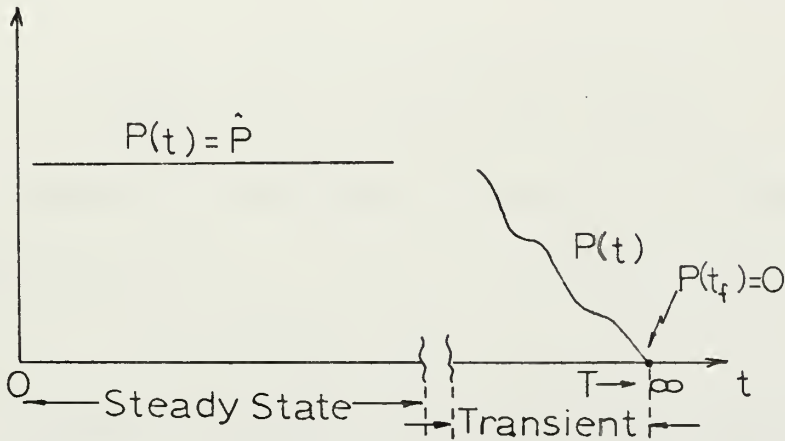


Figure 3.12. Graphical Interpretation of the Solution of Equation 3.67.

It is important to note that $\tilde{P}(t)$ is independent of the states and can be computed once J is specified, before the optimal system is started.

Once $u^*(t)_{\text{opt}}$ is obtained from Eq. 3.60, the final result $u(t)_{\text{opt}}$ is given by $u^*(t)_{\text{opt}} + u_{\text{ss}}$.

b. Kalman's Equation Method of Solution

The foregoing method could be applied without the need for reducing the state equations to the diagonal representation. This is not the case in the present paragraph, because the nice results derived from Kalman's equation, and explained in deep detail in Chapters VII and VIII of Ref. 60, are much more easily used for the diagonal form.

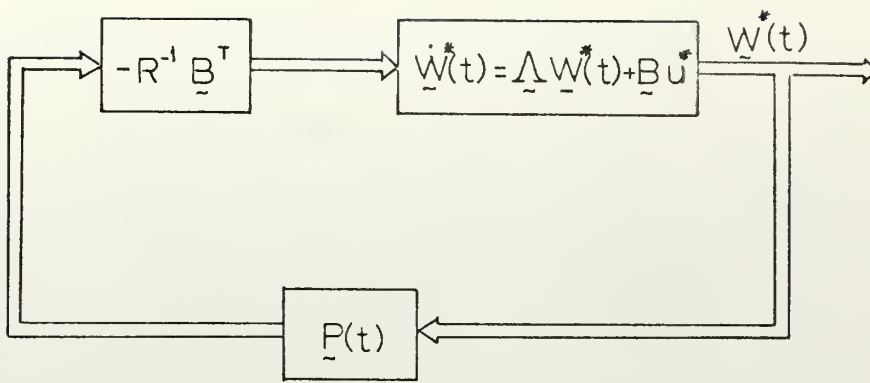


Figure 3.13. The Global System

Suppose an arbitrary \underline{i} row of an \underline{n} dimensional state equation:

$$\dot{w}_i = \lambda_i w_i + b_i u . \quad (3.64)$$

Taking the Laplace transform and ignoring the initial conditions for the sake of clarity, the distributed system can be represented as

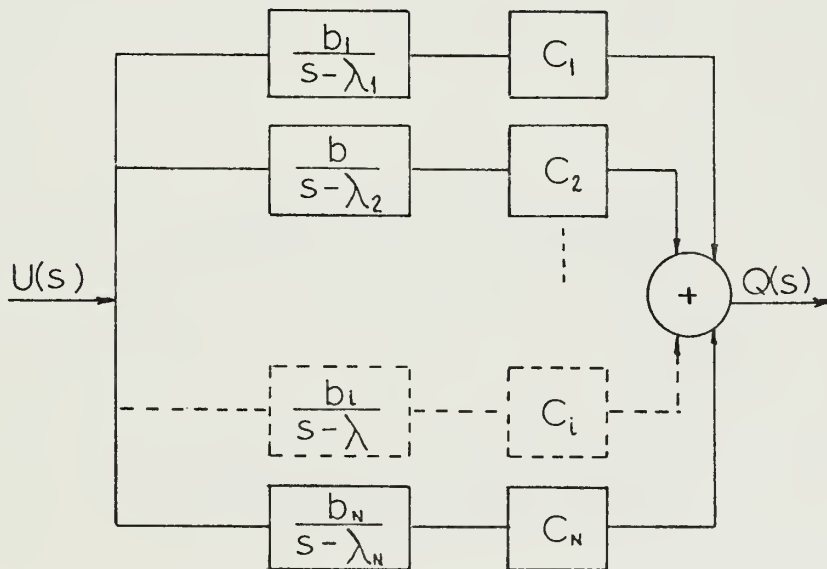


Figure 3.14. Approximate D.P.S. Transfer Function
(becomes exact as $n \rightarrow \infty$)

$$G(s) = \frac{Q^*(s)}{U^*(s)} = \frac{a_{N-1}s^{N-1} + a_{N-2}s^{N-2} + \dots + a_0}{(s-\lambda_1)(s-\lambda_2)\dots(s-\lambda_N)} \quad (3.65)$$

But, as shown in Ref. 60, from Eq. 3.66 also known as Kalman's equation it can be derived that

$$|1 + G(s)H_{eq}(s)|^2 = 1 + \left| \frac{G(s)}{\sqrt{R}} \right|^2 \quad (3.66)$$

where

$$\begin{aligned} |1 + G(s)H_{eq}(s)|^2 &= (1 + G(s)H_{eq}(s))(1 + G(-s)H_{eq}(-s)) \\ |G(s)|^2 &= G(s)G(-s) \end{aligned} \quad (3.67)$$

and that the optimal control is given by

$$U_{opt}^*(s) = r - hW^*(s) \quad (r=0 \text{ in the present case}) \quad (3.68)$$

such that

$$H_{eq}(s) = \frac{hW^*(s)}{Q^*(s)} = \frac{h_1w_1^*(s) + h_2w_2^*(s) + \dots + h_Nw_N^*(s)}{c_1w_1^*(s) + c_2w_2^*(s) + \dots + c_Nw_N^*(s)} \quad (3.69)$$

This is easily illustrated in Figs. 3.15a and 3.15b.

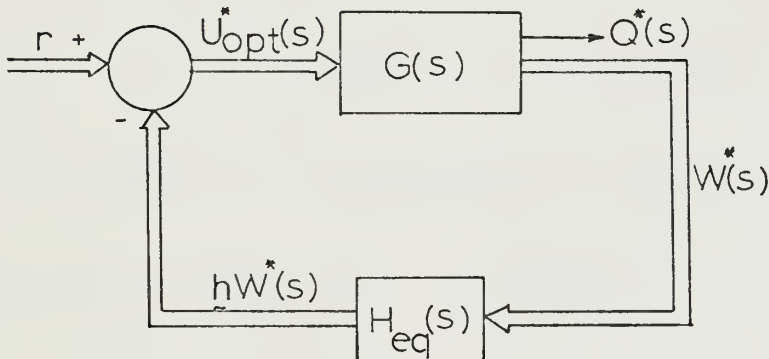


Figure 3.15a. Block Diagram of the Compensated System ($r=0$).

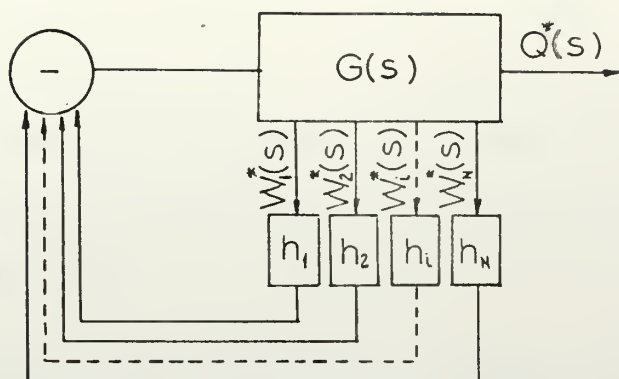


Figure 3.15b. Another Configuration of Figure 3.15a

In order to obtain the elements of \tilde{h} , solve Eq. 3.66 or equivalently, the equation¹³

$$1 + G(s)H_{eq}(s) = \left[1 + \frac{G(s)G(-s)}{R} \right]^+ \quad (3.70)$$

where the $[]^+$ ($[]^-$) indicates the poles and zeros of $1 + \frac{G(s)G(-s)}{R}$ that lie in the left (right) half s plane.

The denominators of both sides of Eq. 3.70 must be equal and the numerator of the left-hand side can be expressed in terms of the h_i 's. Therefore, it is only necessary to find the L.H.P. roots of $\left[1 + \frac{G(s)G(-s)}{R} \right]$, either by

$$1 + \left| \frac{G(s)}{\sqrt{R}} \right|^2 = \left[1 + \frac{G(s)G(-s)}{R} \right]^+ \left[1 + \frac{G(s)G(-s)}{R} \right]^-$$

From Eqs. 3.70 and 3.71 it is possible to write

$$[1 + G(s)H_{eq}(s)][1 + G(-s)H_{eq}(-s)] = \left[1 + \frac{G(s)G(-s)}{R} \right]^+ \left[1 + \frac{G(s)G(-s)}{R} \right]^-$$

Assuming that $G(s)$ has poles only on the L.H.P. it can be shown that the poles of $(1 + G(s)H_{eq}(s))$ are also on the L.H.P.

and therefore $1 + G(s)H_{eq}(s) = \left[1 + \frac{G(s)G(-s)}{R} \right]^+$

root-locus or factorization techniques, after which the coefficients of equal powers of s are equated.

The results obtained by Graham in two given problems, by this method and Riccatti's equation method, are in very good agreement.

D. SHORT COMMENTS ON OBSERVABILITY, STABILITY, CONTROLLABILITY, NONLINEAR PROBLEMS, STOCHASTIC CONTROL AND PARAMETER IDENTIFICATION

The purpose of this paragraph is to give some references to the topics enumerated in the title.

Because Ref. 52 furnishes all the necessary information about the references prior to the end of 1969, only those published after this date will be considered.

1. About Observability and Controllability

Not much has been published in this field, and so far as it is known, all the reports dealing with the observability problem are already mentioned and superficially analyzed in Ref. 52.

Also the controllability problems have been in a period of abandon until recently (1970) when the International Journal of Control published a paper by Herget [30], written in 1968. The author considered a distributed parameter control problem of a linear system with constrained control, either distributed or at the boundary.

The system studied is represented by the equation

$$\frac{\partial Q(\underline{x}, t)}{\partial t} = A(\underline{x})Q(\underline{x}, t) + f(\underline{x})u(t), \quad \underline{x} \in \Omega, \quad t \in T \quad (3.71)$$



with $U(\underline{x})Q(\underline{x},t) = 0$ for \underline{x} in the boundary, and $Q(\underline{x},0) = 0$ for $\underline{x} \in \Omega$.

In the above equations $A(\underline{x})$ represents a partial differential operator defined relatively to Ω , $U(\underline{x})$ for boundary \underline{x} is an operator denoting the boundary conditions and $f(\underline{x})$ is a function also defined in Ω .

The paper shows what are the reachable states of the above type of system for two classes of problems, namely with and without magnitude constraint in the norm of the control ($\|u\|^2$).

2. The Stability Problem

In the field of stability, some recent papers seem to be of great interest because of the wide variety of systems to which they apply.

The following references are mentioned here:

(i) Kathri [Ref. 35] uses multiple Laplace transform techniques in partial differential equations which can be written in the form

$$\sum_{i=0}^I a_i \frac{\partial^i v}{\partial t^i} + \sum_{k=1}^K \sum_{j=1}^J b_{k,j} \frac{\partial^{k+j} v}{\partial t^k \partial x^j} + \sum_{m=1}^M C_m \frac{\partial^m v}{\partial x^m} = u(x,t) \quad (3.72)$$

for a large class of boundary and initial conditions.

The system was reduced to a transfer function expression to which a simple stability criteria was applied. This report is quite comprehensive in the sense that the author furnishes a step by step explanation of all the details.

Later, the same author published a new paper [Ref. 36] in which he considered the same type of systems as before but

applied to sampled data problems where a memory-less, non-linear feedback was present.

(ii) Ansari [Ref. 2] studied the sufficient condition for stability of a system described by

$$\frac{\partial Q(x,t)}{\partial t} + u(t) \frac{\partial Q(x,t)}{\partial x} = -H(u)T^q(x,t) ; \quad q=1,3,5--- \quad (3.73)$$

for $0 < x < L$, $H > 0$ and $u > 0$, where the control $u = f(T(L,t))$ was such that $\frac{H(u)}{u}$ and $Q(0,t)f(Q(L,t))$ are continuous and monotonically decreasing functions of u and $T(L,t)$ respectively. This condition can be verified very easily and although applicable only to a small class of p.d.e.'s, englobes some important diffusions systems.

(iii) D'Souza [Ref. 14] treated a much more general problem than those mentioned previously. He considered a class of causal dynamic systems represented by nonlinear vector-matrix p.d.e.'s. For a good understanding of the developments carried out on the given reference, a good background in Functional Analysis is necessary, with special emphasis in Green's function type of problems. Because such theory is not yet within the range of the knowledge of the average engineer, it seems to be useful to rewrite the paper in more practical terms, if a larger audience is desired.

(iv) Kastenbergl [Ref. 37] derived the stability conditions for nonlinear feedback control systems described by parabolic p.d.e. of the following type

$$\frac{\partial Q(\tilde{x},t)}{\partial t} = \sum_{i,j}^M a_{ij}(\tilde{x}) \frac{\partial^2 Q(\tilde{x},t)}{\partial x_i \partial x_j} + \sum_i^M b_i(\tilde{x}) \frac{\partial Q(\tilde{x},t)}{\partial x_i} + C(\tilde{x}) + F(\tilde{x},t,Q) \quad (3.74)$$

where $F(\underline{x}, t, Q)$ represents the feedback control law, which must be negative definite. The initial and boundary conditions are

$$Q(\underline{x}, 0) = Q_0(\underline{x}) \text{ for } \underline{x} \text{ inside its region of definition and}$$

$$Q(\underline{x}, t) = 0 \quad \text{for } \underline{x} \text{ in the boundary region}$$

the type of system studied is of wide application and the results derived seem to be easy to implement.

3. Nonlinear Systems

Although published in 1966, the paper by Uzgiris and D'Souza, "Optimal Control of Distributed Parameter Systems with Nonlinear Boundary Conditions" [Ref. 67] did not receive much attention. This reason, together with the fact that the procedure developed is relatively simple compared with the complexity of the problems it can solve, motivated the present paragraph.

The authors considered the case of a one-dimensional linear heat equation with nonlinear boundary conditions which were made to include the bounded control function. Then, by discretizing the space variable, a set of nonlinear differential equations was obtained and reduced by Laplace transform and convolution techniques to a nonlinear vector integral equation from which the optimal control could be derived. Following standard methods, the final problem was reduced to the solution of a Volterra type equation which can be solved by known procedures.

4. Stochastic Problems

This paragraph considers the recent papers by Tzafestas [Ref. 66] and Seinfeld, Gavalas and Hwang [Ref. 61].

The first one deals with the identification of parameters of D.P.S.'s by reducing the system to a lumped parameter system for which there exists known parameter identification methods. The last paper contains the derivation of a nonlinear filter for systems with disturbances in the initial and boundary conditions.

5. Identification of Parameters

The present chapter is concluded by mentioning the modern method developed by Fairman and Shen [Ref. 15] in the identification of D.P.S. This method, named by the authors as the "moment functional method" is applicable to one-dimensional heat or wave equations and in the case of the heat equation it may be extended to the problem in which one coefficient is a polynomial function of time.

IV. MODAL CONTROL THEORY

The theory of modal control was first developed by Rosenbrock [Ref. 53] for lumped parameter systems and the concept generalized by Murray-Lasso [Refs. 25 and 45], and Gould [Refs. 25 and 26] in order to include D.P.S.'s.

The above references give the general theory of what is going to be done in the rest of this thesis. This theory, although applicable in many situations, suffers from several drawbacks. The major ones are the need for computing the eigenvalues and eigenvectors of the system's operator, calculation of the inverse of singular matrices, need for \underline{r} controlling elements if \underline{r} modes are to be changed, difficulty of application in the case of repeated eigenvalues and also limitations when the modes are complex conjugates, in which case only the real parts can be compensated.

Some of the inconvenient parts just mentioned were avoided by Simon [Ref. 62] and Foster [Ref. 18], and they furnish very interesting research material in the optimal control field.

A. GENERALITIES

In Fig. 1.1 is shown how a pointwise input produces a distributed output. The idea of the modal control consists in using the measurements at different points, uncoupling these measurements and obtaining the coefficients ω_i of the

eigenfunction expansion of the output.¹⁴ Then, compensate these eigenfunction's coefficients, compare the compensated values with the reference signal and build with this information the adequate control law. This is done in a synthesis procedure conceptually identical to the analysis procedure that gave the coefficients ω_i . The global picture of the control system for a heating problem is indicated in Fig. 4.1 and it shows a basic assumption that was made: The bandlimitedness¹⁵ of the three functions considered here, namely

¹⁴ It is shown in Ref. 45 that in the case of a system described by an operator L such that the output is $y(x,t) = Lm(x,t)$ and

- a) L^{-1} exists,
- b) The eigenfunctions of L are products of functions of time and distance,
- c) L is completely continuous in x , that is to say it has a purely discrete spectrum (eigenvalues content) and the only possible cluster point is the origin, then it can be stated, using the concept of generalized Fourier series, that

$$y(x,t) = \sum_i \omega_i(t) u_i(x)$$

$$m(x,t) = \sum_i \mu_i(t) u_i(x)$$

where $u_i(x)$ is the i^{th} eigenfunction of L , not necessarily sine or cosine functions, and forms a complete set.

Also the operator L is such that its adjoint, L^* , has a complete set of eigenfunctions $v_i(x)$, which is orthogonal to $u_i(x)$.

Using the orthogonality properties in the first of the above equations and assuming normalized distance and eigenfunctions it follows that

$$\omega_i(t) = \int_0^1 y(x,t) v_i(x) dx$$

¹⁵ By bandlimitedness it is meant that the functions are considered accurately described by N eigenfunctions, instead of the infinite number that exactly characterizes all the partial differential equations.

reference temperature, output temperature and control function.

B. LUMPED PARAMETER SYSTEMS

The basic idea in the modal control of lumped parameter systems is the independent shifting of the lower order eigenvalues such that the speed of response of the system is increased.

In the case of an uncontrolled linear system such as

$$\dot{\tilde{x}} = \tilde{A} \tilde{x} \quad (4.1)$$

it is possible to obtain \tilde{x} as a linear combination of terms containing the eigenvectors and eigenvalues of matrix \tilde{A} . For this the transformation

$$\tilde{y} = \tilde{H} \tilde{x} \quad (4.2)$$

is made, from which it follows that

$$\dot{\tilde{y}} = \tilde{H} \tilde{A} \tilde{H}^{-1} \tilde{y} \quad (4.3)$$

If the eigenvalues of \tilde{A} are distinct it is possible to obtain

$$\tilde{H} \tilde{A} \tilde{H}^{-1} = \tilde{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_N \end{bmatrix} \quad (4.4)$$

and therefore

$$y_i = \alpha_i e^{\lambda_i t} \quad (4.5)$$

where $\alpha_i = y_i(0)$.

Finally, from Eq. 4.2 [Ref. 46, p. 112]

$$\tilde{x} = \sum_{i=1}^N \alpha_i \tilde{u}_i e^{\lambda_i t} \quad (4.6)$$

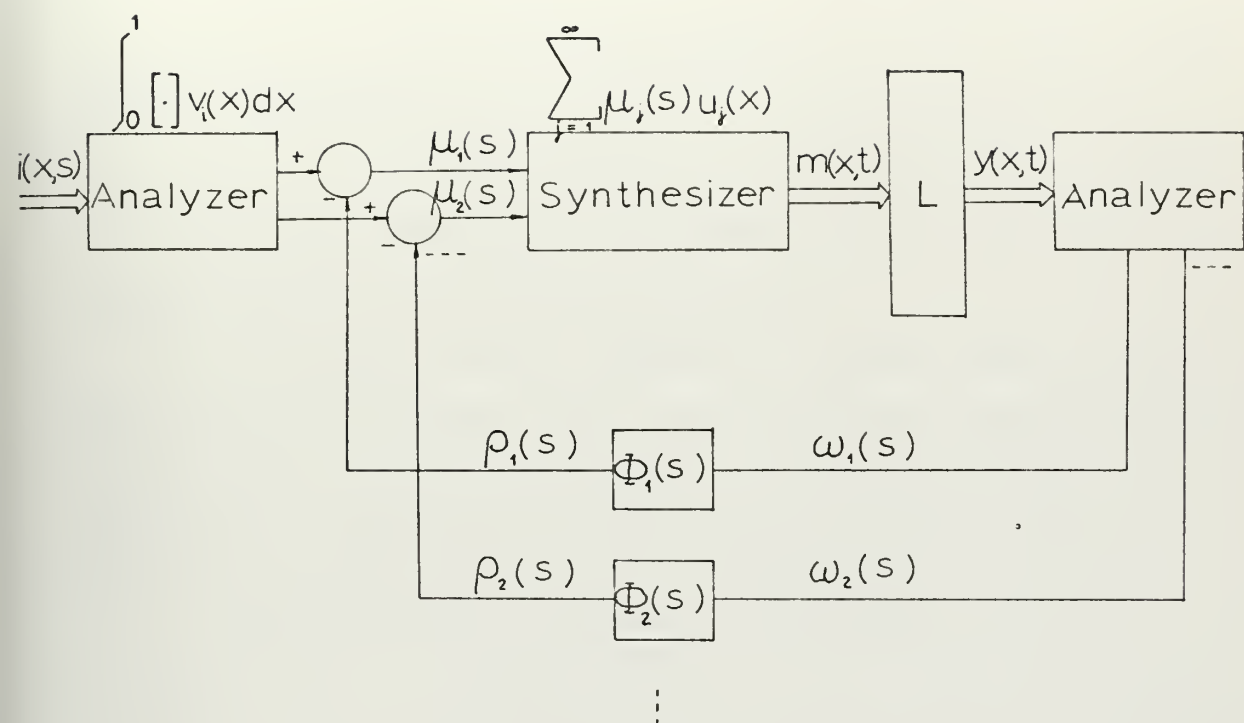


Figure 4.1. Theoretical Implementation of the Modal Control of a D.P.S.

Here u_i is the i^{th} eigenvector of \tilde{A} and λ_i is negative if the system is stable.

The purpose of the modal control is to shift the eigenvalues in such a way that finally the following relationship is obtained

$$\tilde{x} = \sum_{i=1}^H a_i u_i e^{(\lambda_i + k_i)t} \quad (4.7)$$

with k_i a negative real number and a_i the distribution of the initial state into the i^{th} mode. The way this result was obtained is explained next.

1. Ideal Case

The system is described by

$$\dot{\tilde{x}} = \tilde{A} \tilde{x} + \tilde{B} u \quad (4.8)$$

and the output is

$$\tilde{y} = \tilde{C} \tilde{x} \quad (4.9)$$

In this ideal case the dimensions of \tilde{x} and u are N and the dimensions of \tilde{A} and \tilde{B} are $N \times N$.

From basic Linear Algebra it is known that in the case of square matrices with simple and real eigenvalues the eigenvectors of \tilde{A} are orthogonal to the eigenvectors of \tilde{A}^T and both matrices have the same eigenvalues.

From the above statement it follows

$$\begin{aligned} \tilde{A} \tilde{u}_i &= \lambda_i \tilde{u}_i \\ \tilde{A}^T \tilde{v}_i &= \lambda_i \tilde{v}_i \end{aligned} \quad (4.10)$$

and representing now $\tilde{U} = [\tilde{u}_1 \ \tilde{u}_2 \ \dots \ \tilde{u}_N]$ and $\tilde{V} = \begin{bmatrix} \tilde{v}_1^T \\ \tilde{v}_2^T \\ \vdots \\ \tilde{v}_N^T \end{bmatrix}$

the result is

$$\begin{aligned} \tilde{A} \tilde{U} &= \tilde{U} \tilde{\Lambda} \\ \tilde{V} \tilde{A} &= \tilde{\Lambda} \tilde{V}. \end{aligned} \quad (4.11)$$

When the different \tilde{u}_i 's and \tilde{v}_i 's are normalized it can be written

$$\tilde{V} \tilde{U} = \tilde{I} \quad (4.12)$$

and from this equation and the precedent set it is obtained

$$\tilde{A} = \tilde{U} \tilde{\Lambda} \tilde{V}. \quad (4.13)$$

Writing now the state equations as indicated in

Fig. 4.3

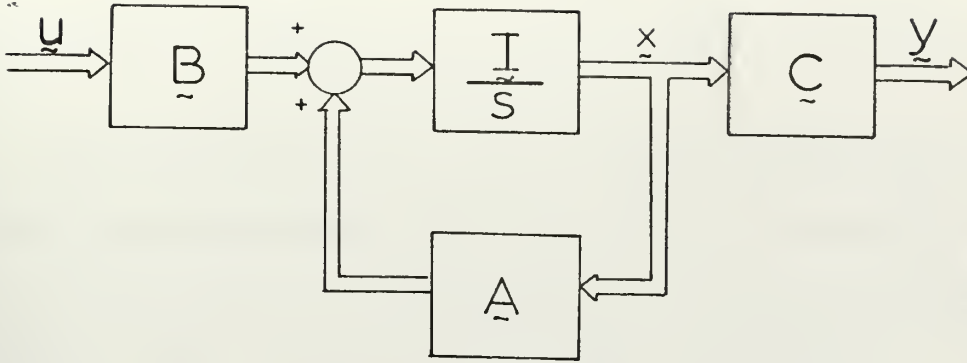


Figure 4.2. Block Representation of an Uncontrolled Linear System.

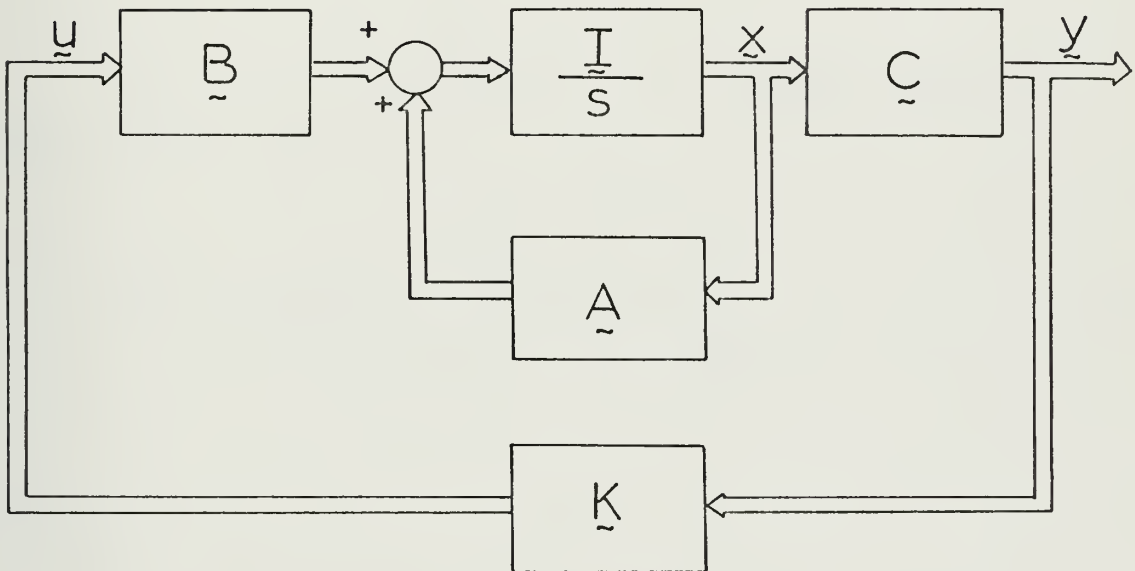


Figure 4.3. Block Representation of a Controlled Linear System.

$$\dot{\tilde{x}} = (\tilde{A} + \tilde{B} \tilde{K} \tilde{C}) \tilde{x} \quad (4.14)$$

Finally, because of Eq. 4.13, a convenient choice for \tilde{B} and \tilde{C} would be making them equal to \tilde{U} and \tilde{V} , which gives

$$\dot{\tilde{x}} = \tilde{U} (\tilde{\Lambda} + \tilde{K}) \tilde{V} \tilde{x} \quad (4.15)$$

The resultant matrix has the same eigenvectors as \tilde{A} ¹⁶ and the eigenvalues have been increased as was desired. Moreover, the shifting of the eigenvalues is realized without interaction, due to the diagonal form of $\tilde{\Lambda}$ and \tilde{K} .

2. The Number of Manipulators (r)¹⁷ Is Less than the Order of the System (N)

Having chosen $\tilde{C} = \tilde{V}$ in Eq. 4.14 the same equation may be written as

$$\dot{\tilde{x}} = (\tilde{U} \tilde{\Lambda} + \tilde{B} \tilde{K}) \tilde{V} \tilde{x} \quad (4.16)$$

Now, in order to have \tilde{B} , a $N \times r$ matrix, again as a square matrix, make

$$\tilde{B} \tilde{K} = \begin{matrix} & \leftarrow r \rightarrow & \leftarrow N-r \rightarrow \\ \begin{matrix} \tilde{U}_r & | & 0 \\ \hline \tilde{U}_{N-r} & | & \tilde{K}_{N-r} \end{matrix} & \begin{matrix} \tilde{K}_r & | & 0 \\ \hline 0 & | & \tilde{K}_{N-r} \end{matrix} & \begin{matrix} \uparrow \\ r \\ \downarrow \\ \uparrow \\ N-r \\ \downarrow \end{matrix} \end{matrix} \quad (4.17)$$

$$\tilde{A} \tilde{u} = \lambda \tilde{u} \quad ^{16}$$

Suppose the eigenvectors are different and call them \tilde{w} . Therefore,

$$\tilde{U}(\tilde{\Lambda} + \tilde{K}) \tilde{V} \tilde{w} = (\tilde{\Lambda} + \tilde{K}) \tilde{w}$$

$$\tilde{A} \tilde{w} + \tilde{K} \tilde{w} = \tilde{\Lambda} \tilde{w} + \tilde{K} \tilde{w}$$

But, the eigenvectors of \tilde{A} are unique and so $\tilde{w} = \tilde{u}$.

¹⁷

A manipulator is the device to which the control law is applied. Its output is the physical input to the system (e.g., heat flow).

Similarly,

$$\begin{matrix} \leftarrow r \rightarrow & \leftarrow N-r \rightarrow \\ \tilde{U} \tilde{\Lambda} = [\tilde{U}_r \mid \tilde{U}_{N-r}] \begin{bmatrix} \tilde{\Lambda}_r & \mid & 0 \\ \hline 0 & \mid & \tilde{\Lambda}_{N-r} \end{bmatrix} \begin{matrix} \uparrow \\ r \\ \downarrow \\ \uparrow \\ N-r \\ \downarrow \end{matrix} \end{matrix} \quad (4.18)$$

and from these three last equations

$$\dot{\tilde{x}} = \tilde{U} \begin{bmatrix} \tilde{\Lambda}_r + K_r & \mid & 0 \\ \hline 0 & \mid & \tilde{\Lambda}_{N-r} \end{bmatrix} \tilde{V} \tilde{x} \quad (4.19)$$

which shows how it is possible to shift arbitrarily the r lowest eigenvalues.

Reference 26, pp. 246-248, contains the theoretical development for the situation in which the eigenvectors $\tilde{u}_1, \dots, \tilde{u}_r$ are not known accurately.

The procedure consists in making

$$\tilde{B}_r = \tilde{D}(\tilde{V}_r \tilde{D})^{-1} = [\tilde{b}_1 \tilde{b}_2 \dots \tilde{b}_r] \quad (4.20)$$

and the final results are similar to the ones obtained previously, with the exception that disturbances in the lower r modes cause also disturbances in the higher $N-r$ modes. This is not always a serious drawback since the influence of the higher modes can generally be neglected.

3. The Number of Sensors (m) is Less than the Order of the System (N)

The derivation for this case is also in Ref. 26. It is a little long and does not follow exactly the same lines as the distributed parameter case. For this reason this paragraph will only be concerned with the basic idea of the process.

Start forming an Nx m matrix

$$\tilde{W} = [\tilde{u}_1 \dots \tilde{u}_r \tilde{u}_{r+1} \dots \tilde{u}_p \tilde{b}_1 \dots \tilde{b}_q] \quad (4.21)$$

where p is the total number of modes taken in account, $q = m - p$, and b is defined as in Eq. 4.20.

Then, form a Nx m matrix \tilde{E} such that \tilde{e}_i ($i=1, \dots, p$) approximates as closely as possible of \tilde{u}_i ($i=1, \dots, p$) and of $\tilde{b}_i - \tilde{u}_i$ for the last q elements \tilde{e}_i of \tilde{E} .

Define \tilde{F} , nx m , such that

$$\tilde{F}^T = (\tilde{E}^T \tilde{W})^{-1} \tilde{E}^T \quad (4.22)$$

This matrix may be written as

$$\begin{aligned} \tilde{F} &= [\tilde{f}_1 \dots \tilde{f}_r \tilde{f}_{r+1} \dots \tilde{f}_p \tilde{f}_1^* \dots \tilde{f}_q^*] \\ &= [\tilde{F}_q | \tilde{F}_{r-q} | \tilde{F}_{p-r} | \tilde{F}_q^*] \end{aligned} \quad (4.23)$$

and from it obtain \tilde{C} defined as

$$\tilde{C}^T = \begin{bmatrix} \overleftarrow{q} & \overleftarrow{r-q} & \uparrow \\ \tilde{F}_q^* + \tilde{F}_q^* & \tilde{F}_{r-q} & \tilde{F}_q^* \\ \downarrow & & \end{bmatrix} \quad (4.24)$$

which has similar properties to \tilde{C} as used in the ideal case. The major difference is that the rows of \tilde{C} are no longer orthogonal to the eigenvectors \tilde{u}_i ($i=p+1, \dots, N$), corresponding to the neglected modes. Therefore, the control of the lower modes interacts with the higher ones not allowing a perfect uncoupling. Generally, under the assumption that the higher modes are sufficiently high, the interaction will be small.

Figure 4.4 shows the block diagram illustrating the present situation. Next it will be explained how the foregoing concepts can be extended to distributed parameter systems.

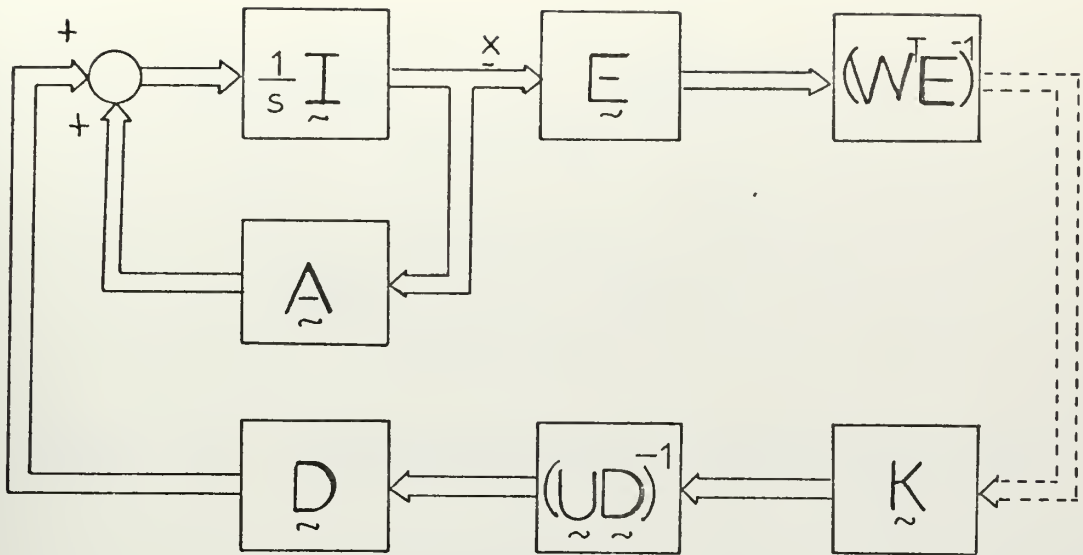


Figure 4.4. Block Diagram when the Number of Sensors Is Less than the Order of the System

C. DISTRIBUTED PARAMETER SYSTEMS

Before starting reading this section one should become familiar with the contents of Appendix B.

1. Computation of the Feedback Control Law

Following lines similar to those that lead to Eq. B.9 it results that in the case of m and y being both functions of x and t , the diagonalization procedure gives

$$\omega_j(s) = \lambda_j(s) \mu_j(s) \quad (4.25)$$

where $\omega_j(s)$ and $\mu_j(s)$ are the Fourier coefficients of $y(x,s)$ and $m(x,s)$ and $\lambda_j(s)$ are the eigenvalues of the operator.

The truncation of the higher order modes allows to describe the control system in matrix form as shown in Fig. 4.5.

Calling \tilde{W} the closed-loop matrix and using the properties of matrix algebra and of uncoupled matrices it can be written:

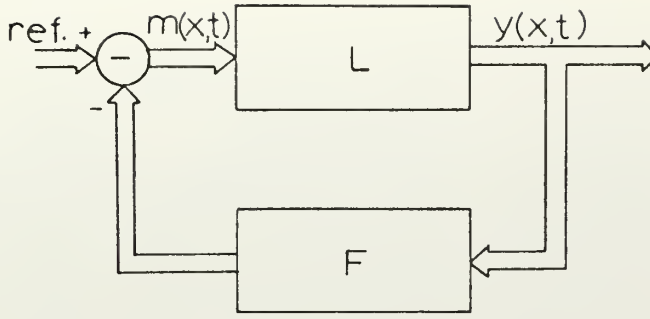


Figure 4.5. Block Diagram of the Closed-Loop System

$$\tilde{W} = (\tilde{L}\tilde{F} + \tilde{I})^{-1} \tilde{L} \quad (4.26)$$

From this it follows

$$\tilde{W}\tilde{L}^{-1} = (\tilde{L}\tilde{F} + \tilde{I})^{-1} \quad (4.27)$$

$$\tilde{L}\tilde{W}^{-1} = \tilde{L}\tilde{F} + \tilde{I} \quad (4.28)$$

and finally

$$\tilde{W}^{-1} = \tilde{F} + \tilde{L}^{-1} \quad (4.29)$$

The system can then be represented as in Fig. 4.6 and for each loop the result is

$$\phi_n(s) = \frac{1}{w_n(s)} - \frac{1}{\lambda_n(s)} \quad (4-30)$$

where the $\phi_n(s)$, $w_n(s)$ and $\lambda_n(s)$ are, respectively, the diagonal elements of \tilde{F} , \tilde{W} and \tilde{L} . By a suitable choice of the ϕ_n 's it is therefore possible to make each closed loop behave as fast as required.

The situation illustrated in Fig. 4.6 although desirable is not possible in practice because of the existence of sensors restricted to certain positions and of manipulators which cannot give exactly the desired output distribution. It will be shown next that in a physical system it is possible to

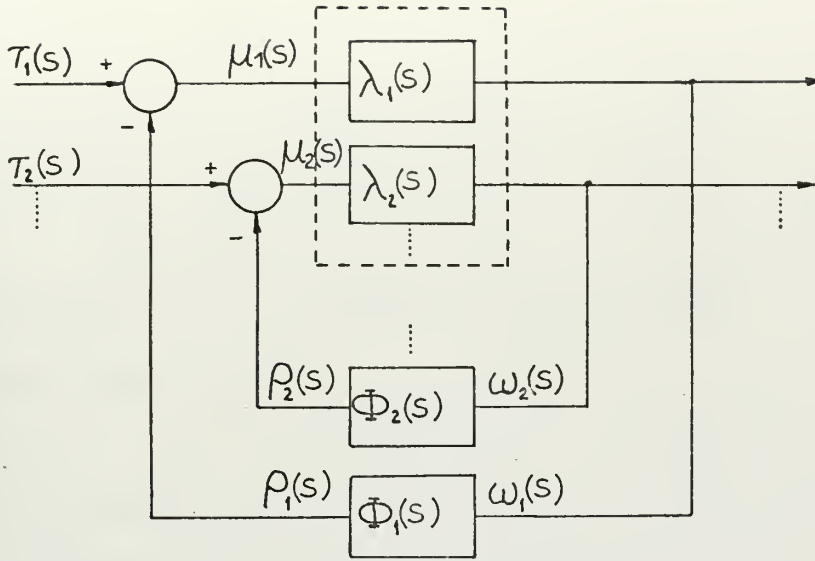


Figure 4.6. The Uncoupled System

obtain the coefficients ω_i from the output measurements, and α_i , the actual input to each manipulator, from the knowledge of the distribution produced by the manipulators.

2. Determination of the Fourier Coefficients of the Output from the Output Measurements

Under the assumption of bandlimitedness

$$y(x, t) = \sum_{i=1}^N \omega_i(t) u_i(x) \quad (4.31)$$

Similarly, the measurements at x_1, x_2, \dots, x_N may now be written as

$$\begin{aligned} y(x_1, t) &= \sum_{i=1}^N \omega_i(t) u_i(x_1) \\ &\vdots \\ y(x_N, t) &= \sum_{i=1}^N \omega_i(t) u_i(x_N) \end{aligned} \quad (4.32)$$

or, in matrix form

$$\tilde{Y} = \tilde{U} \tilde{\Omega} \quad (4.33)$$

where

$$\tilde{Y} = \begin{bmatrix} y(x_1, t) \\ y(x_2, t) \\ \vdots \\ y(x_N, t) \end{bmatrix}, \quad \tilde{U} = \begin{bmatrix} u_1(x_1) u_2(x_1) \dots u_N(x_1) \\ u_1(x_2) u_2(x_2) \dots u_N(x_2) \\ \vdots \\ u_1(x_N) u_2(x_N) \dots u_N(x_N) \end{bmatrix}, \quad \tilde{\Omega} = \begin{bmatrix} \omega_1(t) \\ \omega_2(t) \\ \vdots \\ \omega_N(t) \end{bmatrix} \quad (4.34)$$

It follows that

$$\tilde{\Omega} = \tilde{U}^{-1} \tilde{Y} \quad (4.35)$$

and it is proved in Ref. 45, Appendix C, that there is always a set of points x_1, x_2, \dots, x_N such that \tilde{U}^{-1} exists. It was verified in the simulation procedure of the present work that even in the case of the sensors being relatively close, it was possible to reconstruct accurately the coefficients ω_i . The only differences are in having higher values of the elements in \tilde{U}^{-1} and in the sensitivity to errors in the measurements. There are several criteria described in Refs. 18 and 45 for the minimization of the sensitivity. The one used consists in the minimization of the maximum eigenvalue of the product.

$$(\tilde{U}^\dagger)^T \tilde{U}^\dagger \quad (4.36)$$

and is valid whenever the number of sensors (S) is greater or equal than the order of the approximated system (N).

The symbol (\dagger) represents a pseudo-inverse matrix and it will be defined in paragraph d. If $S=N$, the pseudo-inverse coincides with the inverse.

When $S < N$ the sensors must be positioned according to the minimization of the norm $\| \tilde{I} - \tilde{U}^\dagger \tilde{U}(x_i) \|$, where now \tilde{U}

is function only of one element of the set $\{x_i\}$. \tilde{U}^\dagger is also a different matrix and it can be obtained as explained in Ref. 18, pp. 60-63. The minimization is equivalent to obtaining the minimum value of the maximum eigenvalue of $|\tilde{I} - \tilde{U}^\dagger \tilde{U}(x_i)|$.

3. Determination of the effect of the manipulators

Call H the distribution caused by the i^{th} manipulator and decompose it in an eigenfunction expansion.

$$\begin{aligned} H_1(x) &= b_{11}u_1(x) + b_{12}u_2(x) + \dots + b_{1N}u_N(x) \\ &\vdots \\ H_N(x) &= b_{N1}u_1(x) + b_{N2}u_2(x) + \dots + b_{NN}u_N(x) \end{aligned} \quad (4.37)$$

The instantaneous output of the i^{th} manipulator (assuming instantaneous response) for the actual input $\alpha_i(t)$ is therefore

$$\alpha_i(t) H_i(x) \quad (4.38)$$

and given that the desired input is

$$m(x,t) = \sum_{j=1}^N \mu_j(t) u_j(x). \quad (4.39)$$

Equating both

$$\sum_{i=1}^N \alpha_i(t) H_i(x) = \sum_{j=1}^N \mu_j(t) u_j(x) = \sum_{i=1}^N \alpha_i \sum_{k=1}^N u_k(x) b_{ik} \quad (4.40)$$

or

$$\tilde{B}^T \tilde{\alpha} = \tilde{\mu} \quad (4.41)$$

and $\tilde{\alpha}$ can be obtained as

$$\tilde{\alpha} = (\tilde{B}^T)^{-1} \tilde{\mu} \quad (4.42)$$

The problem of finding $(\tilde{B}^T)^{-1}$ is not a trivial one and for many types of manipulators it does not exist.

One way of checking for the existence of the inverse is to compute the determinant of B ; if its value is very small it is better to try to avoid it because a tremendously high control gain will be required.

By far the best situation is when the distributions of the manipulators coincide with the eigenfunctions, in which case B will be the identity matrix. This situation is not very realistic; however, it is frequently possible to move the position of the manipulators. In this case, the best obtainable positioning (η_i) turns out to be the one which maximizes the projection of $H_i(x-\eta_i)$ on $u_i(x)$.

The adequate choice of the manipulators is also a question of good sense, and before starting to move them and trying to maximize the above mentioned projection, it is highly desirable to look physically at the distribution and see if there is any possibility of obtaining it with the available manipulators. As an example, it may be said that given a system described by Eq. 2.7 it would be absurd to try to get the heat configuration shown in Fig. 4.7 with the heat source positioned as shown.

It is now possible to represent the control of the given system in an easily implementable way, as illustrated in Fig. 4.8.

It should be mentioned that the coefficients τ_j are not the Fourier coefficients of the desired output distribution ($\tau_j(t) = \int_0^1 i(x,t) v_j(x) dx$), but these values multiplied

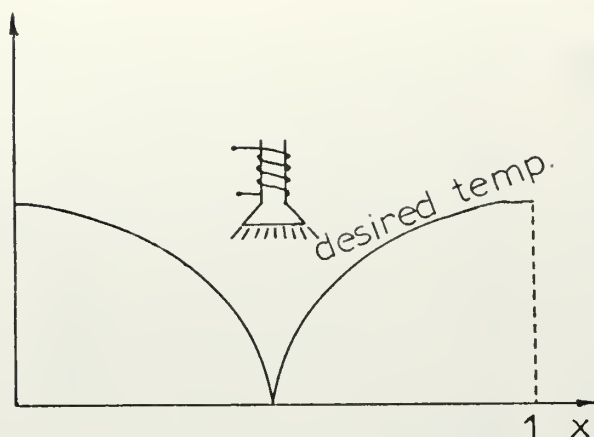


Figure 4.7. Example of an Undesired Control

by $\frac{1+\phi_j \lambda_j G_j}{\lambda_j G_j}$, the inverse of the closed-loop gain;¹⁸

G_j represents the gain of the j^{th} manipulator. This statement is also not always exact as will be seen later in the analysis of the computer results. Meanwhile it may be added that in many cases a minimum square error deviation may not be desired, but another kind of criteria; in such conditions the τ 's must be chosen using an optimizing program, as for example the gradient search. When the number of filters is too high this search probably will require a great deal of computer time and it may become difficult to determine if the obtained optimum is local or absolute. This procedure was implemented successfully for three reference coefficients and one hundred

¹⁸ The product of the coefficients τ_j by the inverse of closed-loop gain will be mentioned throughout the rest of this thesis as the "compensated Fourier coefficients."

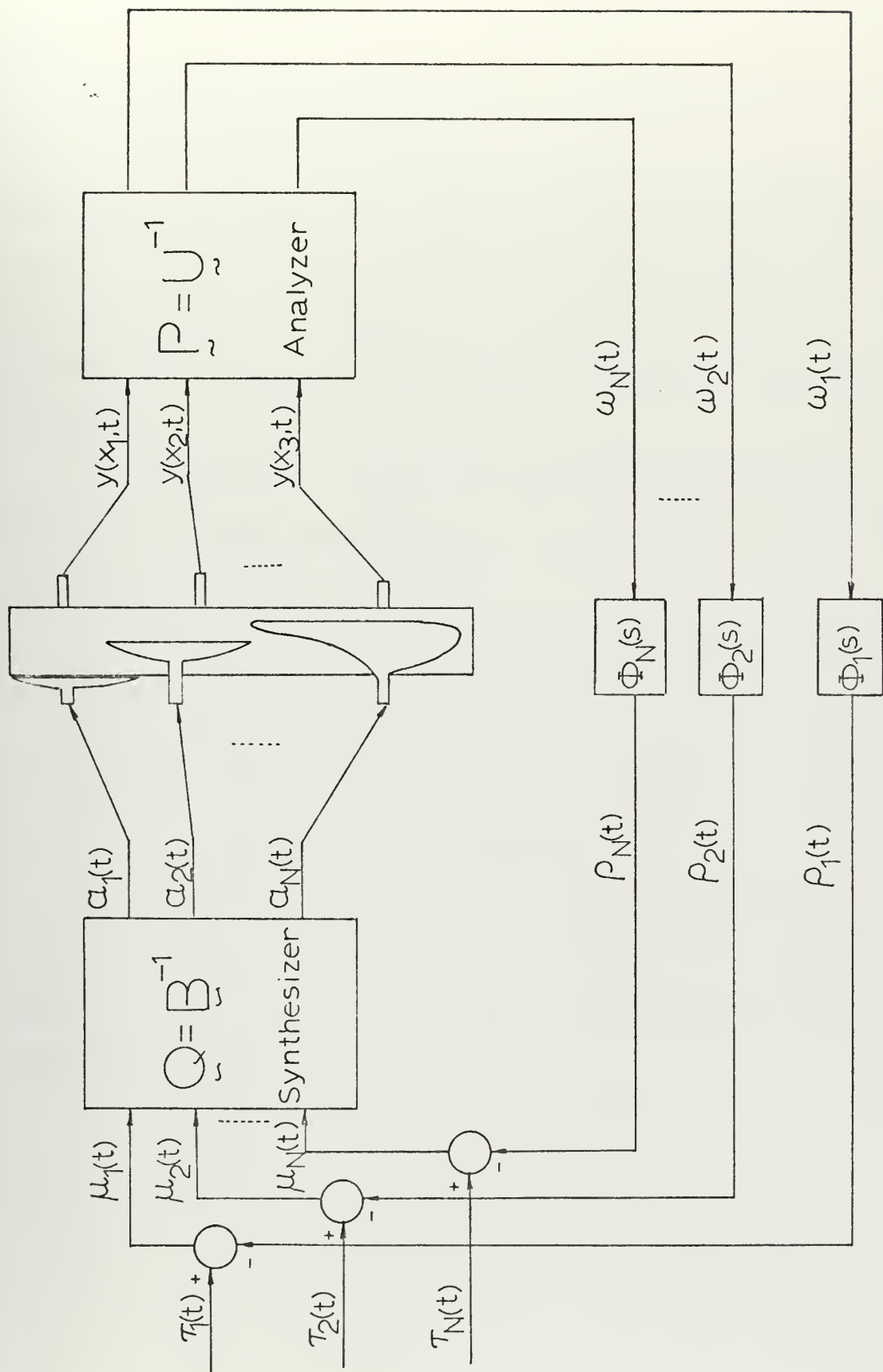


Figure 4.8. The Model of the System

solutions of the system only took two minutes of computer time.

4. Practical Limitations

As in the lumped parameter case the most common situation takes place when the order of the approximated system differs from the number of sensors or manipulators. In this case the problem of inverting non-square matrices must be considered and it can be solved using the concept of the pseudo-inverse (\downarrow).

The rules for calculation of the pseudo inverse of real matrices are simply synthesized as follows. For further information see Refs. 42 and 73.

- a. If \tilde{A} is in diagonal form, \tilde{A}^\downarrow is also diagonal, with the non-zero elements the reciprocal of those of \tilde{A} and with zeros where \tilde{A} has zeros.
- b. Knowing that $\tilde{A}^T \tilde{A}$ is a square matrix obtain \tilde{A}^\downarrow as

$$\tilde{A}^\downarrow = (\tilde{A}^T \tilde{A})^{-1} \tilde{A}^T = \tilde{A}^T (\tilde{A} \tilde{A}^T)^{-1} \quad (4.43)$$

The computer implementation of these concepts is shown in the next chapter. Depending on the numerical values of the elements in the matrices sometimes one procedure is better than the other one.

V. DETAILED SOLUTION OF A PROBLEM BY MODAL CONTROL TECHNIQUES

A. PROBLEM DESCRIPTION

It is desired to heat a given rod at three different points such that a certain temperature distribution is achieved.

Geometrically, Fig. 5.1 shows the system configuration. Looking at one differential volume element it is possible to derive the equations describing the process.

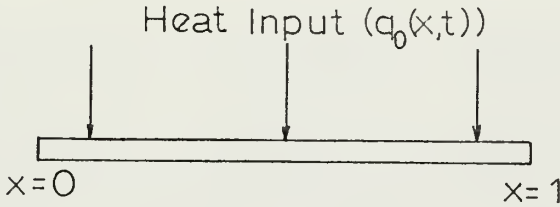


Figure 5.1. The Heating of a Rod

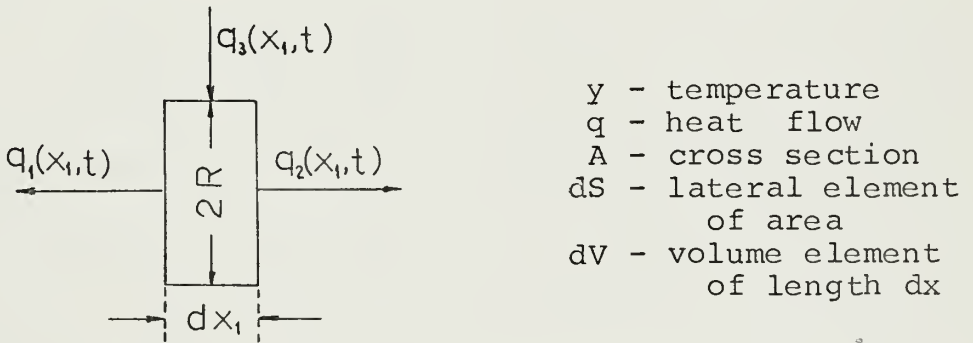


Figure 5.2. Heat Flux in a Volume Element

Consider the principle of conservation of energy; from Fig. 5.2 the following equation can be written:

$$(q_1 - q_2)A + q_3 dS = \frac{\partial E}{\partial t_1} dV \quad (5.1)$$

However,

$$q_2(x_1) = q_1(x_1+dx_1) \approx q_1(x_1) + \frac{\partial q_1(x_1)}{\partial x} dx_1 \quad (5.2)$$

and this gives

$$q_1 - q_2 = - \frac{\partial q_1}{\partial x_1} dx_1 \quad (5.3)$$

Taking $E = c\rho y_1$ where c is the specific heat at constant volume and ρ the mass density it follows

$$- \frac{\partial q_1}{\partial x_1} A dx_1 + q_3 S dx_1 = c\rho \frac{\partial y_1}{\partial t_1} A dx_1 \quad (5.4)$$

or

$$- \frac{\partial}{\partial x_1} (-k \frac{\partial y_1}{\partial x_1}) \pi R^2 dx_1 + q_3 2\pi R dx_1 = c\rho \frac{\partial y_1}{\partial t_1} \pi R^2 dx_1 \quad (5.5)$$

where k is the thermal conductivity (Btu/sec-ft-°F). Eliminating the common terms and taking $\alpha = k/\rho c$ (α is the thermal diffusivity):

$$\alpha \frac{\partial^2 y_1}{\partial x_1^2} + \frac{2q_3}{\rho c R} = \frac{\partial y_1}{\partial t_1} \quad (5.6)$$

This equation can be normalized and simulated, taking into account the scaling factors, as

$$\frac{\partial^2 y}{\partial x^2} + q = \frac{\partial y}{\partial t}, \quad (0 \leq x \leq 1, t \geq 0) \quad (5.7)$$

For simplicity the following initial and boundary conditions are considered:

$$y(x, 0) = 0, \quad 0 \leq x \leq 1 \quad (5.8)$$

$$\left. \begin{aligned} y(0, t) &= 0 \\ \frac{\partial y}{\partial x}(1, t) &= 0 \end{aligned} \right\} \quad 0 \leq t < \infty \quad (5.9)$$

Assume the availability of two different sets of three manipulators, one giving a heat distribution coincident with the eigenfunctions, as shown in Fig. 5.3, and the other giving the heat distribution illustrated in Fig. 5.4.

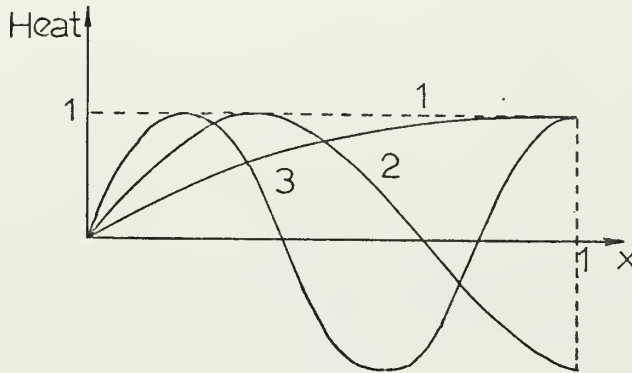


Figure 5.3. Heat Distribution No. 1

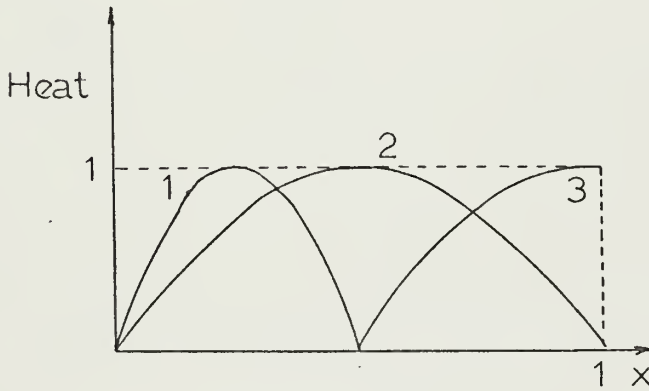


Figure 5.4. Heat Distribution No. 2

Throughout the simulation runs only the two desired temperature distributions of Figs. 5.5 and 5.6 will be considered, the first one of which has the shape of the first eigenfunction and being for this reason easily attainable.

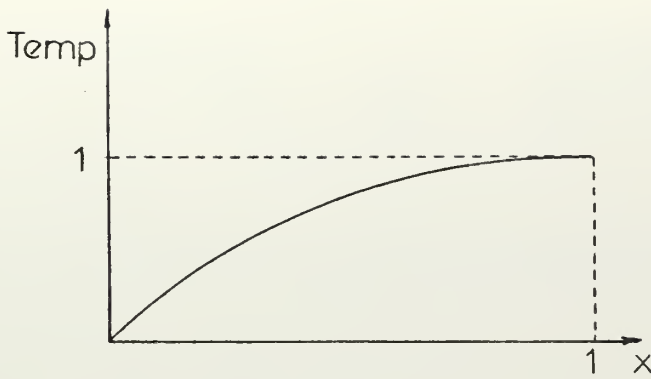


Figure 5.5. Temperature Distribution No. 1

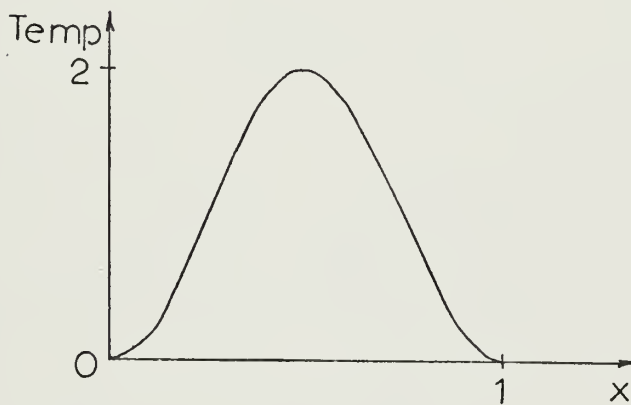


Figure 5.6. Temperature Distribution No. 2

B. DERIVATION OF THE EIGENFUNCTIONS AND EIGENVALUES

Consider again Fig. B.1. By Laplace transforming the distance dependent operator with respect to t it follows that:

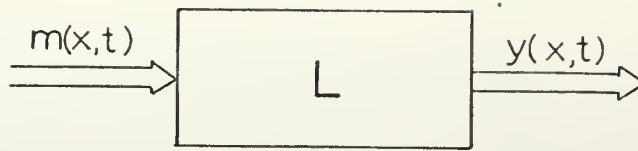


Figure B.1. Representation of the System in Operator Form

$$M[\cdot] = L^{-1}[\cdot] = \begin{cases} [-\frac{d^2}{dx^2} + s]\{\cdot\} \\ T(0,s) = 0; \frac{dT(x,s)}{dx}\bigg|_{x=1} = 0 \end{cases} \quad (5.10)$$

To find the eigenfunctions make

$$M u = \lambda u \quad (5.11)$$

$$\text{or} \quad \begin{cases} -\frac{d^2 u}{dx^2} + (s-\lambda)u = 0 \\ u(0,s) = 0, \frac{du(x,s)}{dx}\bigg|_{x=1} = 0 \end{cases} \quad (5.12)$$

whose solution is

$$u = K_1 e^{j\sqrt{\lambda-s}} + K_2 e^{-j\sqrt{\lambda-s}x} \quad (5.13)$$

From the boundary condition $u(0,s)=0$:

$$u(x,s) = C \sin\sqrt{\lambda-s} x \quad (5.14)$$

and, from the other boundary condition,

$$\sqrt{\lambda_n - s} = (2n+1)\frac{\pi}{2} \quad (5.15)$$

or

$$\lambda_n = s + (2n+1)^2 \frac{\pi^2}{4} \quad (5.16)$$

In order to normalize the eigenfunctions take $\langle u_n, u_n \rangle = 1$ or

$$\int_0^1 u_n^2(x,s) dx = 1. \quad \text{The following expression results:}$$

$$C_n^2 \int_0^1 \sin^2 \sqrt{\lambda_n - s} \, x \, dx = 1, \quad (5.17)$$

from which $C_n = \sqrt{2}$

The next step is the determination of the adjoint operator, M^+ , by using the common technique indicated in Ref. 45, pp. 269-271 and by Crandall [Ref. 12, p. 211].

By definition of adjoint operator write

$$\langle M u, v \rangle = \langle u, M^+ v \rangle \quad (5.18)$$

Integrating by parts the left-hand side:

$$\begin{aligned} \int_0^1 \left[-\frac{d^2 u}{dx^2} + (s-\lambda)u \right] \cdot v \, dx &= - \left. \frac{du}{dx} \cdot v \right|_0^1 + \int_0^1 \frac{du}{dx} \cdot \frac{dv}{dx} \, dx + \\ &+ \int_0^1 (s-\lambda) \cdot v \, dx = \left[-\frac{du}{dx} \cdot v \right]_0^1 + u_1 \left. \frac{dv}{dx} \right|_0^1 - \int_0^1 u \cdot \frac{d^2 v}{dx^2} + \int_0^1 (s-\lambda) \cdot v \, dx \end{aligned} \quad (5.19)$$

From the equations just above the definition of adjoint operator is verified taking

$$M^+ = -\frac{d^2}{dx^2} + (s-\lambda) \quad (5.20)$$

and

$$v(0) = \left. \frac{dv}{dx} \right|_{x=0} = 0 \quad (5.21)$$

Because $M^+ = M$ and the boundary conditions for M^+ are the same as those for M , the former is said to be self-adjoint. This fact permits writing:

$$M[\cdot] = \sum_{n=1}^{\infty} \lambda_n(s) \, u_n(x) \int_0^1 v_n(\zeta) \, d\zeta \quad (5.22)$$

and, according to an important property of the functions of an operator (see also Eq. B11).

$$f(M[\cdot]) = \sum_{n=1}^{\infty} f(\lambda_n(s)) u_n(x) \int_0^1 v_n(\zeta) [\cdot] d\zeta \quad (5.23)$$

which means that if $f(M[\cdot]) = M^{-1}[\cdot] = L[\cdot]$

it follows that

$$L[\cdot] = \sum_{n=1}^{\infty} \frac{1}{\lambda_n(s)} u_n(x) \int_0^1 v_n(\zeta) [\cdot] d\zeta \quad (5.24)$$

Next, compute the first six eigenvalues from Eq. 5.17:

i	λ_i	$\sqrt{\lambda_i - s}$	
0	$s+2.4674$	1.57080	
1	$s+22.2066$	4.71239	
2	$s+61.6850$	7.85398	(5.25)
3	$s+120.9027$	10.99558	
4	$s+199.8595$	14.13717	
5	$s+298.5555$	17.27876	

The corresponding first three eigenvalues of the operator L are

$$\begin{aligned} \lambda_0(s) &= \frac{1}{s+2.4674} \\ \lambda_1(s) &= \frac{1}{s+22.2066} \\ \lambda_2(s) &= \frac{1}{s+61.6850} \end{aligned} \quad (5.26)$$

C. COMPUTATION OF THE FEEDBACK CONTROL LAW

Suppose it is desired to make the first three eigenvalues coincide with the fourth one. Using Eq. 4.30 it follows:

$$\begin{aligned} \phi_0 &= s + 120.9027 - s - 2.4674 = 118.4353 \\ \phi_1 &= s + 120.9027 - s - 22.2066 = 98.6961 \\ \phi_2 &= s + 120.9027 - s - 61.6850 = 59.2177 \end{aligned} \quad (5.27)$$

D. COMPUTATION OF THE SENSORS' POSITIONS AND OF \tilde{U}^{-1}

Consider now the cases of having three and six sensors.

The second of Eqs. 4.35 is repeated

$$\tilde{U} = \begin{bmatrix} u_1(x_1) & u_2(x_1) & \text{----} & u_N(x_1) \\ u_1(x_2) & u_2(x_2) & \text{----} & u_N(x_2) \\ & \vdots & & \\ u_1(x_S) & u_2(x_S) & \text{----} & u_N(x_S) \end{bmatrix} \tag{4.49}$$

where x_1, x_2, \dots, x_S are the sensors' locations.

Two computer programs were written for the conditions $S \geq N$. The first one divides the interval $x(0,1)$ in ten parts and tries all the possible combinations of x_1, x_2, x_3 within this interval, such that $0 < x_1 < x_2 < x_3 \leq 1$. If the temperature at $x=0$ was not known, the possibility of a sensor there should also be considered. Once the optimal rough positions are computed, this information is introduced in a gradient subroutine which searches for the rigorous optimal positions. Both programs make use of the same function subprogram (EVMAX), which in turn uses two IBM library subroutines (MINV and MPRD, respectively for the inverse and product of matrices) and one N.P.G.S.'s subroutine (JACVAT) for the computation of the eigenvalues of a real-symmetric matrix.

The optimal positions and corresponding U and U^\dagger matrices derived for the cases of three and six sensors are:

CASE I-

$$x(1) \text{ to } x(3) = .28570E00 \quad .57144E00 \quad .85715E00 \tag{5.28}$$

$$\begin{aligned}
U_{11} \text{ to } U_{13} &= .61357E00 \quad .13787E01 \quad .11058E01 \\
U_{21} \text{ to } U_{23} &= .11057E01 \quad .61353E00 \quad -.13788E01 \quad (5.29)
\end{aligned}$$

$$\begin{aligned}
U_{31} \text{ to } U_{33} &= .13788E01 \quad -.11057E01 \quad .61367E00 \\
U_{11}^{\downarrow} \text{ to } U_{13}^{\downarrow} &= .17532E00 \quad .31593E00 \quad .39391E00 \\
U_{21}^{\downarrow} \text{ to } U_{23}^{\downarrow} &= .39393E00 \quad .17533E00 \quad -.31591E00 \quad (5.30) \\
U_{31}^{\downarrow} \text{ to } U_{33}^{\downarrow} &= .31589E00 \quad -.39391E00 \quad .17532E00
\end{aligned}$$

CASE II—

$$\begin{aligned}
x(1) \text{ to } x(3) &= .15406E00 \quad .30828E00 \quad .46234E00 \\
x(4) \text{ to } x(6) &= .61621E00 \quad .76996E00 \quad .92325E00 \quad (5.31)
\end{aligned}$$

$$\begin{aligned}
U_{11} \text{ to } U_{13} &= .338904E00 \quad .938860E00 \quad .132315E01 \\
U_{21} \text{ to } U_{23} &= .658373E00 \quad .140436E01 \quad .932899E00 \\
U_{31} \text{ to } U_{33} &= .939128E00 \quad .116083E01 \quad -.665084E00 \quad (5.32) \\
U_{41} \text{ to } U_{43} &= .116491E01 \quad .333099E00 \quad -.140276E01 \\
U_{51} \text{ to } U_{53} &= .132288E01 \quad -.661515E00 \quad -.330576E00 \\
U_{61} \text{ to } U_{63} &= .140394E01 \quad -.132272E01 \quad .116496E01 \\
U_{11}^{\downarrow} \text{ to } U_{16}^{\downarrow} &= .05225 \quad .10148 \quad .14464 \quad .17915 \quad .20310 \quad .21531 \\
U_{21}^{\downarrow} \text{ to } U_{26}^{\downarrow} &= .14483 \quad .21645 \quad .17871 \quad .05125 \quad -.10151 \quad -.20289 \\
U_{31}^{\downarrow} \text{ to } U_{36}^{\downarrow} &= .20383 \quad .14385 \quad -.10216 \quad .21588 \quad -.05109 \quad .17894 \\
&\quad (5.33)
\end{aligned}$$

E. COMPUTATION OF THE REFERENCE COEFFICIENTS

Using the definition of τ_j write

$$\tau_j(t) = \int_0^1 i(x,t) v_j(x) dx \quad (5.34)$$

where $i(x,t)$ is the desired temperature distribution. This gives, for the two desired temperature distributions shown in Figs. 5.5 and 5.6, respectively

$$\begin{aligned}
 \text{TAU (1)} &= \frac{\sqrt{2}}{2} = 0.707107 \\
 \text{TAU (2)} &= 0.0 \\
 \text{TAU (3)} &= 0.0
 \end{aligned}
 \tag{5.35}$$

and

$$\begin{aligned}
 \text{TAU (1)} &= 0.960317 \\
 \text{TAU (2)} &= 0.685931 \\
 \text{TAU (3)} &= -0.32011
 \end{aligned}
 \tag{5.36}$$

F. DERIVATION OF B FROM THE HEAT DISTRIBUTION OF THE MANIPULATORS

For the heat distribution coincident with the eigenfunctions, \tilde{B} is the identity matrix.

In the case of the heat distribution No. 2, obtain from Eq. 4.38 (using the orthonormality properties) and Fig. 5.4 the coefficients of B as follows:

$$\begin{aligned}
 b_{11} &= \sqrt{2} \int_0^{0.5} \sin 2\pi x \sin \frac{\pi}{2} dx = .169765 \\
 b_{12} &= \sqrt{2} \int_0^{0.5} \sin 2\pi x \sin \frac{3\pi}{2} dx = .363783 \\
 b_{13} &= \sqrt{2} \int_0^{0.5} \sin 2\pi x \sin \frac{5\pi}{2} dx = .353678 \\
 b_{21} &= \sqrt{2} \int_0^1 \sin \pi x \sin \frac{\pi}{2} dx = .600211 \\
 b_{22} &= \sqrt{2} \int_0^1 \sin \pi x \sin \frac{3\pi}{2} dx = .360126 \\
 b_{23} &= \sqrt{2} \int_0^1 \sin \pi x \sin \frac{5\pi}{2} dx = -.085744 \\
 b_{31} &= -\sqrt{2} \int_0^1 \cos \pi x \sin \frac{\pi}{2} dx = .424413 \\
 b_{32} &= -\sqrt{2} \int_0^1 \cos \pi x \sin \frac{3\pi}{2} dx = -.381972 \\
 b_{33} &= -\sqrt{2} \int_0^1 \cos \pi x \sin \frac{5\pi}{2} dx = .151576
 \end{aligned}
 \tag{5.37}$$

After this $\tilde{Q} = (\tilde{B}^T)^{-1}$ must be computed. The result came out to be the following one:

$$\begin{array}{llll} Q(1,1) \text{ to } Q(1,3): & -.12283 & .71648 & 2.14944 \\ Q(2,1) \text{ to } Q(2,3): & 1.07012 & .69963 & -1.23328 \\ Q(3,1) \text{ to } Q(3,3): & .89194 & -1.27602 & .88434 \end{array} \quad (5.38)$$

G. NUMERICAL SOLUTION OF THE PARABOLIC DIFFERENTIAL EQUATION THAT DESCRIBES THE SYSTEM

The system was simulated using the Crank-Nicolson method. The following system of equations resulted, where J1 stands for the instant of time J+1. The normalized distance interval was divided in 20 parts and the time interval was made equal to 1/400.

$$\begin{bmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & -1 & 4 & -1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 4 & -1 \\ & & & & -2 & 4 \end{bmatrix} \begin{bmatrix} u_{2,j1} \\ u_{3,j1} \\ u_{4,j1} \\ \vdots \\ u_{20,j1} \\ u_{21,j1} \end{bmatrix} = \begin{bmatrix} u_{1,j1} \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u_{1,j} \\ u_{2,j} \\ u_{3,j} \\ \vdots \\ u_{19,j} \\ 2u_{20,j} \end{bmatrix} + \begin{bmatrix} u_{3,j} \\ u_{4,j} \\ u_{5,j} \\ \vdots \\ u_{21,j} \\ 0 \end{bmatrix} + 2\delta t \begin{bmatrix} q_{2,j} \\ q_{3,j} \\ q_{4,j} \\ \vdots \\ q_{20,j} \\ q_{21,j} \end{bmatrix} \quad (5.39)$$

This system is solved in double precision by IBM subroutine "DGE1B." In order to take in account the fact that the sensors are not generally in the exact subdivisions of the distance domain, also IBM subroutine "ALI" was used, which interpolates for the sensors' positions.

In Eq. 5.39 the letter u stands for temperature (contrarily to the rest of the chapter where the temperature is represented by y) in order to be consistent with the computer program.

H. SIMULATION RESULTS BASED IN KNOWN THEORETICAL FACTS

Ten different simulations were performed. These simulations correspond to changes in the following parameters: feedback gains, manipulators output shape and gain, reference signal and number of sensors. The runs involving the use of strong gain saturation and a control different from the eigenfunction control gave origin to the development of new practical procedures. Due to the relevancy of such a fact, the next chapter will be entirely devoted to it.

This thesis shows, so far as it is known, the first closed-loop simulation of the modal control of a distributed parameter system by non-optimal control techniques. In order to illustrate the characteristics of this model the present section is dedicated to emphasize the results of changes in the three most representative parameters, namely the feedback gains, the direct gains and the reference coefficients.

With exception of run No. 7 all the other ones were made using the Crank-Nicolson method, as described in Chapter II.C. 1. This is a brute force method that has the advantage that it is possible to make it quite accurate only by reducing the time and distance intervals, but, in turn, it requires a sophisticated computer program and, consequently, a long computer time. The program written makes use of I.B.M. subroutine

"DGELB" for the double precision solution of a system of linear equations, and "ALI" which takes the temperature at the discretized points in the rod and interpolates these values in order to get the temperatures at the sensors' positions. At the end of this thesis is shown the computer program used in run No. 10, which serves as reference program for the other runs.

Run No. 7 was done using the Bubnov-Galerkin transformation and this permitted writing a fairly simple program using a prediction-correction numerical solution of ordinary differential equations [Milne, Ref. 44]. This technique can very easily be applied to a large variety of systems, as described by Foster [18]. However, when dealing with the modal control approach, the matrix \tilde{B} of the Bubnov-Galerkin method cancels the matrix $\tilde{Q} = \tilde{B}^{-1}$ in the feedback loop. This makes the transformation only applicable in the case of having the manipulator distribution coincident with the eigenfunctions, for which \tilde{B} is the identity matrix. For this reason Foster uses the method only in an optimal control way, applied to the linear regulator problem, such that the mentioned transformation does not take place.

1. Run No. 1: Feedback Gain Equal to Unity

The differences from the reference program are:

- a. Dimension - $U(21,200)$
- b. Data - $NT = 200$
- c. $TAU(1) = .707107$
- d. $PHI(1) = PHI(2) = PHI(3) = 1.0$

e. DO 38 ID=1, N1

EV1(ID) = 1.57080*H*(ID-1)

EV2(ID) = 4.71239*H*(ID-1)

EV3(ID) = 7.85398*H*(ID-1)

H1(ID) = 1.414214*SIN(EV1(ID))

H2(ID) = 1.414214*SIN(EV2(ID))

38H3(ID) = 1.414214*SIN(EV3(ID))

f. First IQ is 4, second IQ is 40 and the last is 200

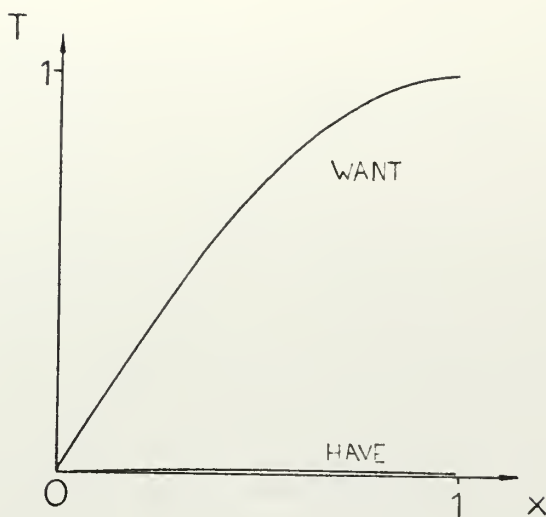
g. Q is the identity matrix

h. The loop "DO 41 ..." was not included and this allowed negative control.

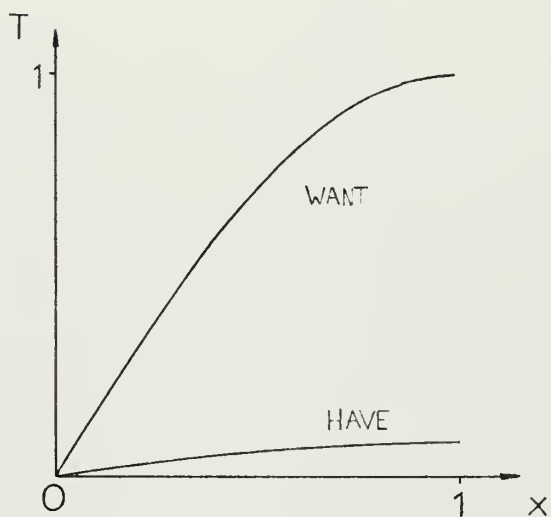
In this run it was intended to obtain the temperature distribution No. 1 (Fig. 5.5) using manipulator control No. 1 (Fig. 5.3); the reference temperature coefficients were not divided by the respective closed-loop gains. As it can be observed the output never approached the desired distribution, converging very slowly toward an amplitude much lower than the desired one; this illustrates the need for using the compensated Fourier coefficients as described in Chapter IV.3.

2. Run No. 2: Increasing the Feedback Gain of the First Eigenfunction

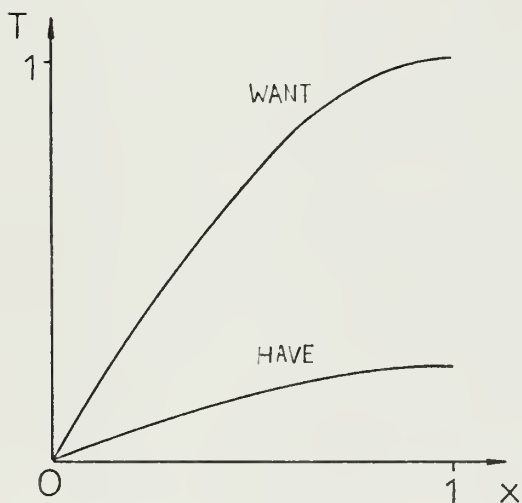
Same statements as in previous run except for $\text{PHI}(1) = 5.0$. The convergence although faster than in the preceding case was still too slow. Notice that the closed-loop gain corresponding to the first eigenfunction decreased. This run illustrates how the total gain of the system decreases when the feedback is increased.



a) $T = 0.01$

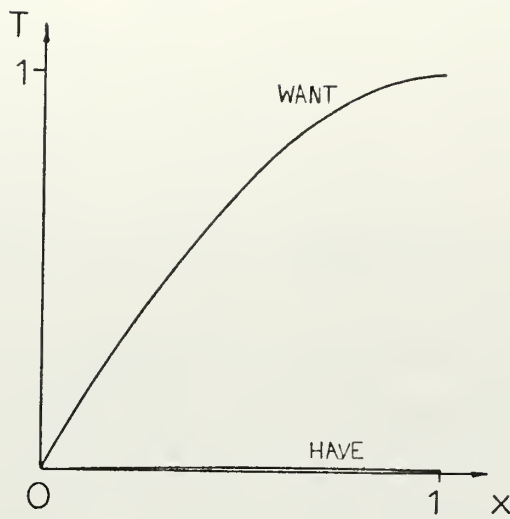


b) $T = 0.1$

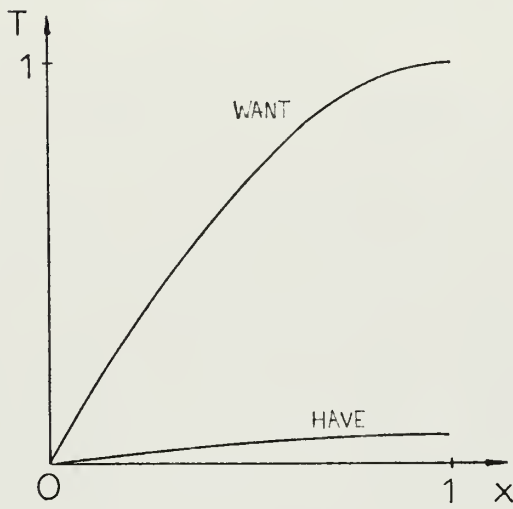


c) $T = 0.5$

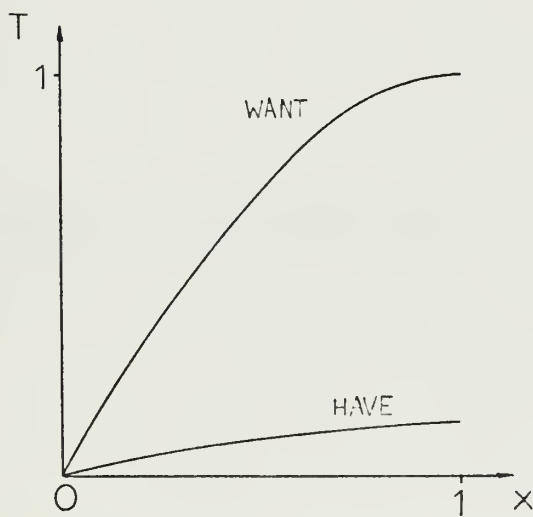
Figure 5.7. Run No. 1



a) $T = 0.01$



b) $T = 0.1$



c) $T = 0.5$

Figure 5.8. Run No. 2

3. Run No. 3: Increase of the Manipulators' Gains

The differences relative to case No. 1 are as follows:

c. $\text{TAU}(1) = 3.62272$

d. $\text{PHI}(1) = 5.0$

f. First IQ is 1, second IQ is 2 and the third one is 40.

Also in statement 43 the total value was multiplied by 20, which is equivalent to an increase of the manipulator's gains by the same factor. The response was very fast and at $T=.02$ secs the system was practically stabilized at the desired temperature distribution.

The indicated value of $\text{TAU}(1)$ was found after having divided the original $\text{TAU}(1)$ by the respective closed-loop gain. The result of this division is 3.62277, practically equal to the 3.62272 that gave the exact desired heat distribution. The reason for this small difference is due to the fact that the matrices \tilde{P} and \tilde{Q} are approximate transformations (in the specific case of eigenfunction control consider only the effects of \tilde{P}) and maybe due to round-off errors. In general, if N is high the empirical and theoretical values must be very close to each other.

4. Run No. 4: Making the First Three Eigenvalues to Coincide with the Fourth One

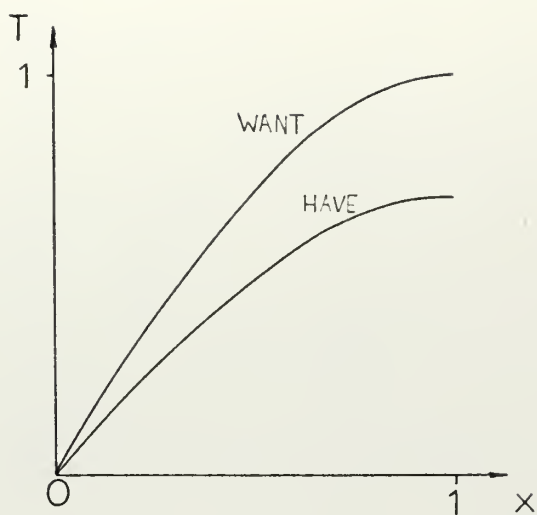
The statements below are the only changes in the last run.

$$\text{PHI}(1) = 118.5353$$

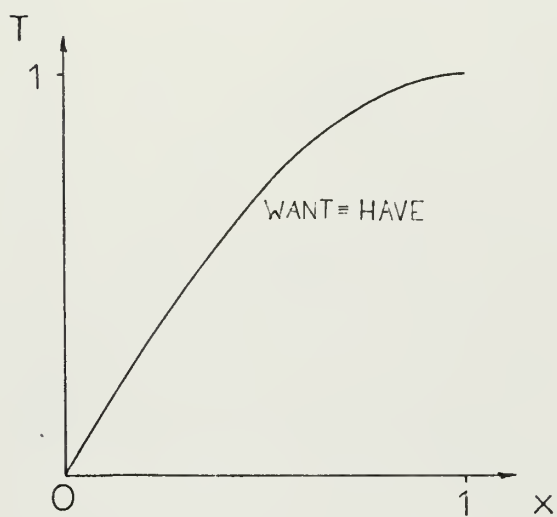
$$\text{PHI}(2) = 98.6961$$

$$\text{PHI}(3) = 59.2177$$

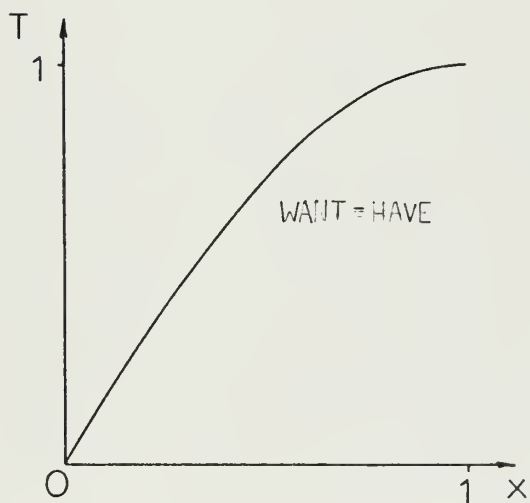
$$\text{TAU}(1) = 85.502 \text{ (derived value was } 85.503)$$



a) $T = 0.01$

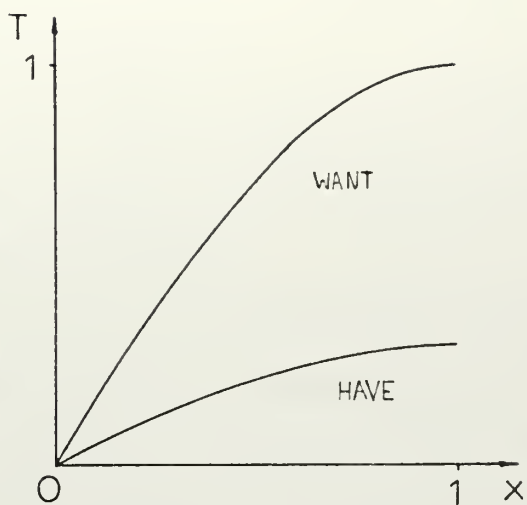


b) $T = 0.1$

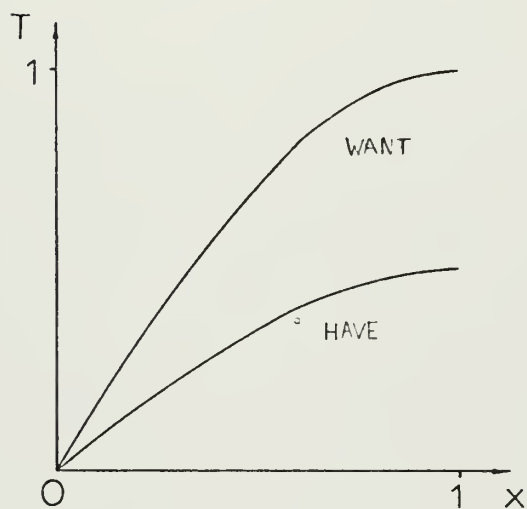


c) $T = 0.5$

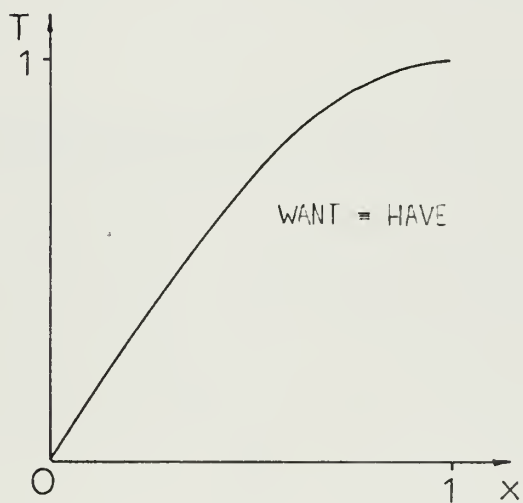
Figure 5.9. Run No. 3



a) $T = 0.0025$



b) $T = 0.005$



c) $T = 0.1$

Figure 5.10. Run No. 4

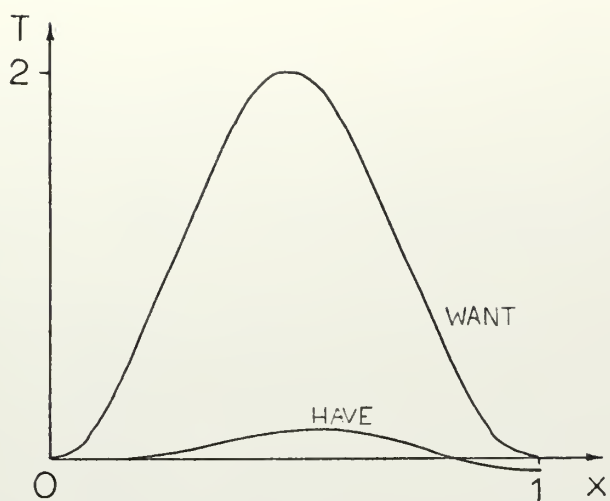
Again statement 43 has the normal gain. The results were practically the same as in the case just considered, with the advantage that much less heat power was required. When trying to increase the gain of the manipulators by a factor of 20 the system became unstable.

Some other modifications were done, as for example $\text{PHI}(2)=\text{PHI}(3)=0$ but no reasonable change was observed, which shows the dominant influence of the first eigenfunction in this specific problem; it is not so in many other cases. Another change was in the position of the sensors; for $x(1)=.3, x(2)=.4$ and $x(3)=.5$ a different corresponding matrix \tilde{P} with much greater values was obtained. As expected, the transformation carried on by \tilde{P} gave the necessary eigenfunction information and the results are practically the same as when the sensors were in the optimal positions (they differ only in the fourth decimal place). The important difference between these cases resides in the fact that the optimal positions are the ones for which the errors in the measurements are minimized.

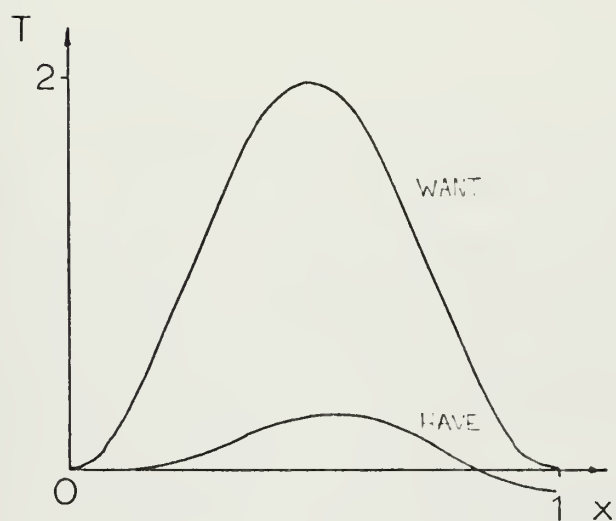
5. Run No. 5: A different Output Distribution and Nonlinear Control

In order to have temperature distribution No. 2 (Fig. 5.6) the last run was modified as follows:

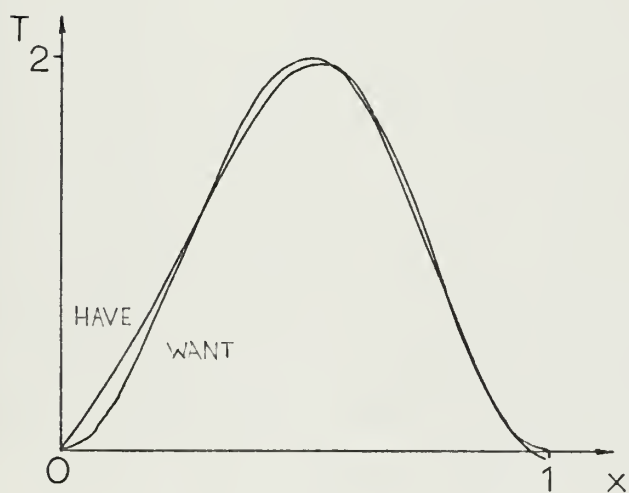
```
90 DO 91 IH=1,N1
X1(IH)=(IH-1)*H
D=6.283185*X1(IH)
Y2(IH)=1.0-COS(D)
91 CONTINUE
```

a) $T = 0.0025$



b) $T = 0.005$



c) $T = 0.1$

Figure 5.11. Run No. 5

A saturated amplifier with $-20 \leq \alpha \leq 20$ was used. It resulted that although this control is quite saturated, a good output (at steady-state it equals the one obtained, with linear control) was obtained with the analytically derived τ 's, namely:

$$\text{TAU}(1) = 116.10$$

$$\text{TAU}(2) = 82.93$$

$$\text{TAU}(3) = -38.70$$

Statement 40 was substituted by the sequence:

```
40 DO 41 I=1,MP
```

```
    IF (ALPHA(I).GT.20.0) ALPHA(I)=20.0
```

```
41 IF (ALPHA(I).LT.-20.0)ALPHA(I)=-20.0
```

6. Run No. 6: Larger Number of Sensors

The number of sensors was increased from three to six. This required, relative to the preceding case, the following changes in the computer program:

```
DIMENSION - x(6),P(3,6),TEMP(6)
```

```
DATA      - KS=6
```

```
x(1) to x(6) as in Eq. 5.31 and  $\underline{p}$  as in Eq. 5.33
```

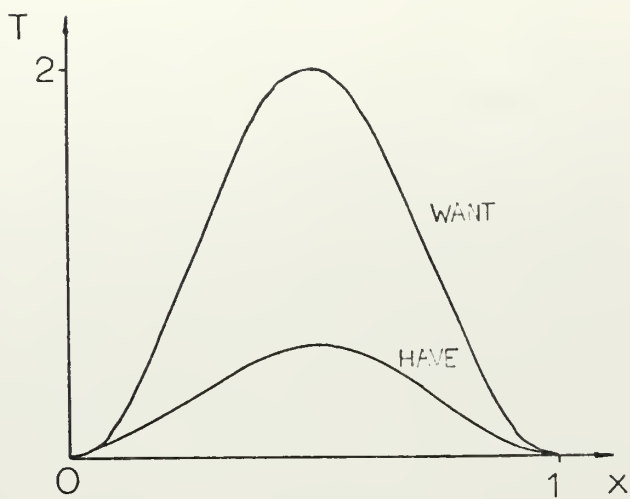
```
TAU(1)=116.10
```

```
TAU(2)= 82.93
```

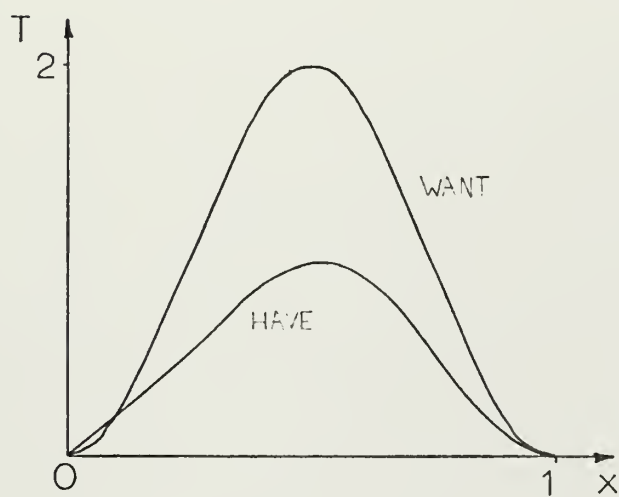
```
TAU(3)=-38.70
```

The loop 40 DO 41 ____etc., was removed and again statement 40 became the same as in the reference program.

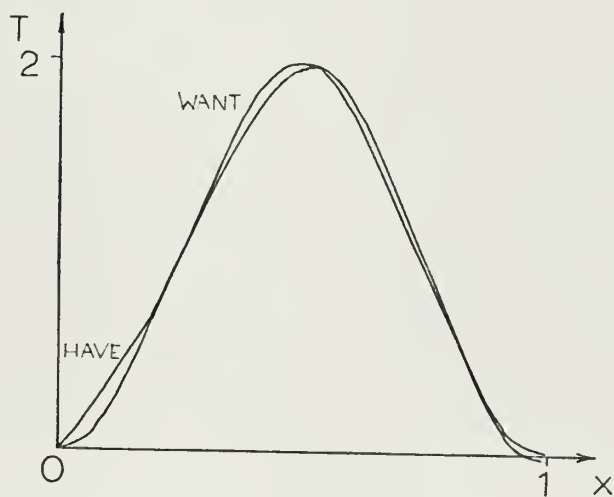
This run was done in order to illustrate the technique of using the pseudo-inverse matrix.



a) $T = 0.0025$



b) $T = 0.005$

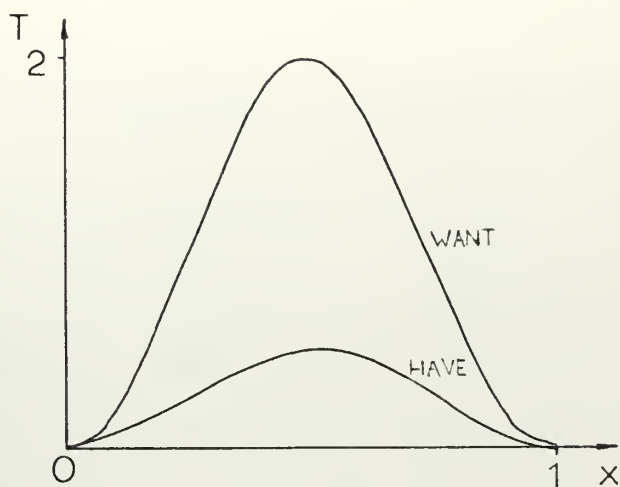


c) $T = 0.1$

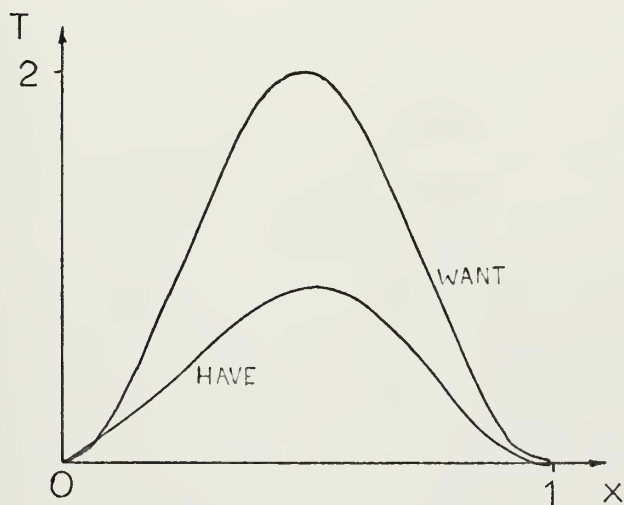
Figure 5.12. Run No. 6

7. Run No. 7: The Bubnov-Galerkin Transformation

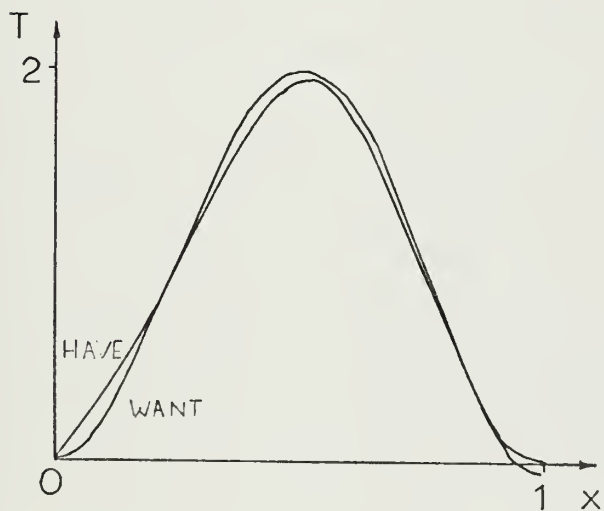
The computer program for this case is also shown in the end of the thesis. The loop elements are exactly as in paragraph five but the nonlinearity was removed. Practically no difference is noticed in steady state, which is far better than expected and shows that in the present example the higher order modes of the system can be neglected.



a) $T = 0.0025$



b) $T = 0.005$



c) $T = 0.1$

Figure 5.13. Run No. 7

VI. SIMULATION RESULTS BASED ON A NEW TECHNIQUE

It is a mathematical fact that the Fourier coefficients of an N terms expansion minimize the square error of the approximation. When the matrices \tilde{P} and \tilde{Q} are high order matrices (theoretically infinite order) the use of the compensated Fourier coefficients for input reference gives exactly the desired output. This may be the reason why Gould states [Ref. 26, p. 242] that typically the number of sensors is of the order of fifty and the number of manipulators of the order of ten. When the analysis described by \tilde{P} is already relatively accurate (this happens in the example given in this thesis even with only a third order matrix) and the synthesis done by \tilde{Q} is also precise (is exact for eigenfunction control) the compensated Fourier coefficients also give origin to an output distribution close to the desired one. For a better understanding of what follows the above examples are named as ideal cases. However, when the accuracy of the transformations defined by \tilde{P} and \tilde{Q} is inadequate it is possible to improve it by changing the reference coefficients under the control of a gradient search until an output distribution that better approaches the desired one is achieved. In order to be able to use the gradient search, similarly to what was done for the computation of the optimal sensors' positions, a certain cost-function (SQERR) was defined.

If it is intended to minimize the square error between the desired and obtained distributions, one convenient definition of SQUERR is, according to Fig. 5.14:

$$\text{SQUERR} = \sum_{i=1}^M (y1_i - y2_i)^2$$

The computer program used to implement such a technique is shown in the end of the thesis.

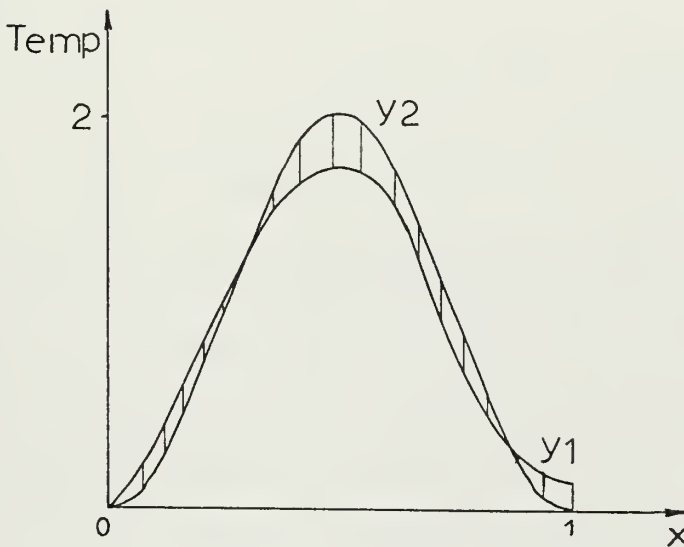


Figure 6.1. How to Define the Cost Function

For the ideal case ($\tilde{Q}=\tilde{I}$ and \tilde{P} an accurate transformation) SQUERR was found to be .0966 and for the non-ideal one worked in run No. 10 ($\tilde{Q}\neq\tilde{I}$ and same \tilde{P} as before) it was computed as .1117. If the compensated Fourier coefficients are used instead of the gradient search in any of the given non-ideal cases, the cost becomes much higher (1.1 in run No. 10). In all the examples worked out M was taken as twenty-one.

Another important feature of the new technique is that it permits, through a convenient weighting of the different terms in the functional cost, the desired output to be fitted in certain zones even more exactly than in the ideal case. One example of this situation is given in the run No. 9 for which SQUERR was defined as

$$\text{SQUERR} = \sum_{i=1}^M k_i (y1_i - y2_i)^2$$

where M is as before and

$$k_i = 1 \text{ for } i=1 \text{ to } 7 \text{ and } 15 \text{ to } 21$$

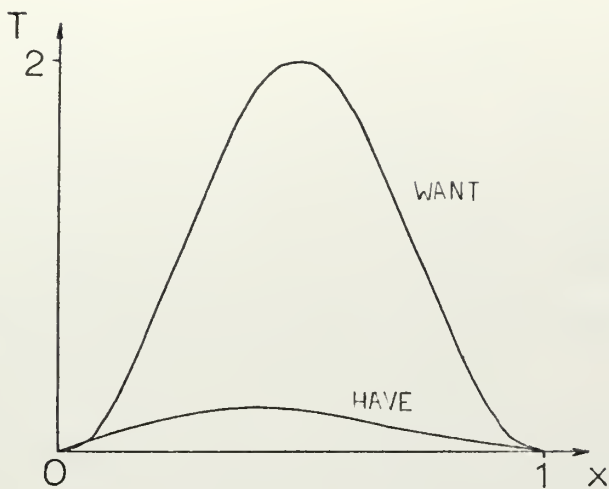
$$k_i = 10 \text{ for } i=8 \text{ to } 10 \text{ and } 12 \text{ to } 14$$

$$k_i = 100 \text{ for } i=11$$

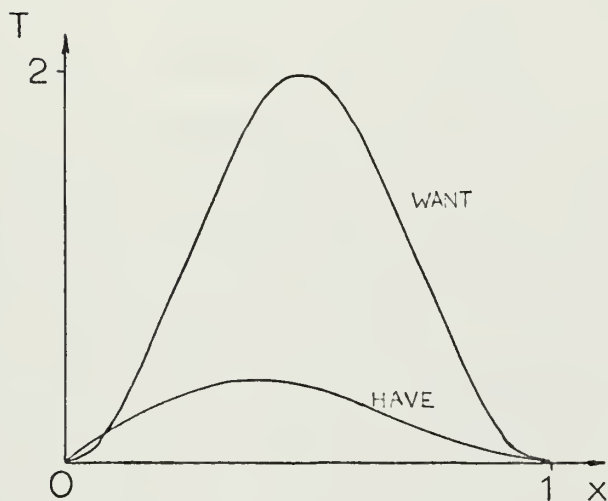
1. Run No. 8 Saturated Amplifier with $0 \leq \alpha \leq 40$

Up to now the signal of the flux given by the manipulators was not restricted. This does not mean that the flux may be negative (removal of heat) but that a change of variable was done in the equation describing the process, such that the simulated heat flow corresponds physically to a higher level heat flow. In some circumstances the mentioned transformation may be undesirable and therefore it becomes necessary to take into account in the simulation the fact that the control cannot go negative. The output was far from the desired one and then it was thought that maybe another set of τ 's would be better in nonlinear cases.

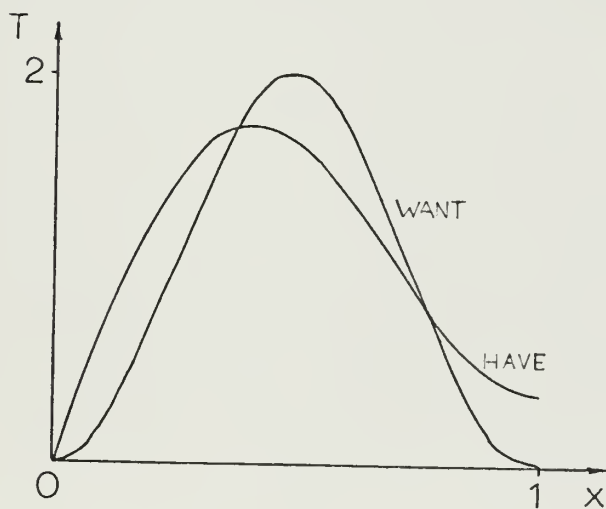
To verify the suppositions just mentioned and knowing that the Fourier coefficients give a minimum square error distribution, a cost function was defined (SQUERR) which was



a) $T = 0.0025$



b) $T = 0.005$



c) $T = 0.1$

Figure 6.2. Run No. 8

the summation of the squares of the differences between the desired and the obtained distributions at every one-twentieth fraction of the system's dimension. The new τ 's did not come out very different but the deviation suggested that in some cases it is possible to have a better output with a set of τ 's different from the Fourier coefficients. This is true, as it will be seen in the next runs.

The optimal set of reference coefficients is

TAU(1) = 114.62

TAU(2) = 84.38

TAU(3) = 38.70

The remaining of the program is as in run No. 5, except for the nonlinearity which is defined as

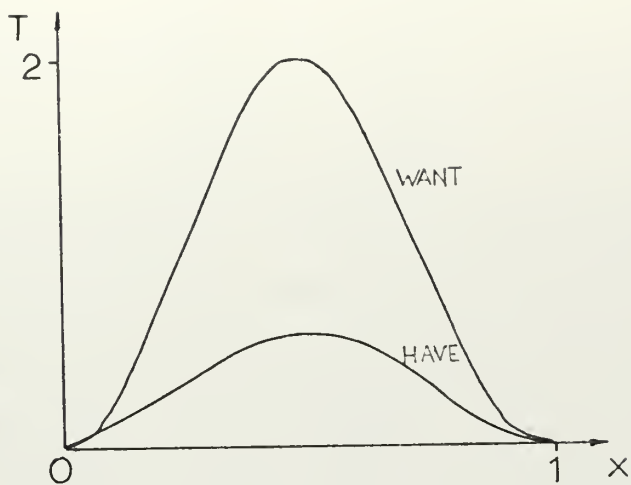
```
40 DO 41 I=1,MP
```

```
IF (ALPHA(I).GT.40.0) ALPHA(I)=40.0
```

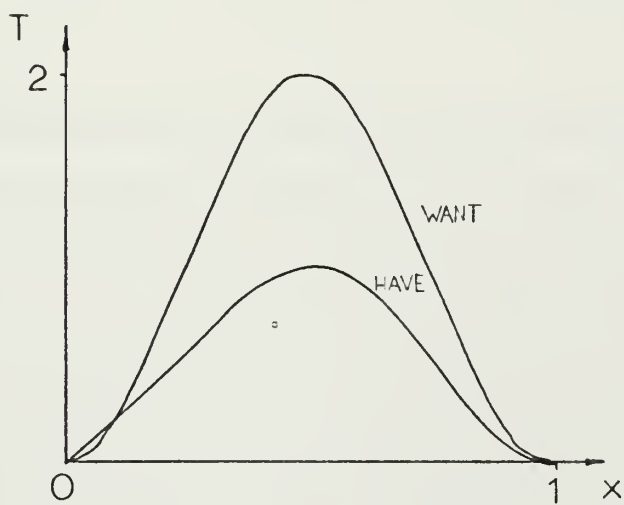
```
41 IF (ALPHA(I).LT.0.0) ALPHA(I)= 0.0
```

2. Run No. 9: A Locally Better Output Distribution

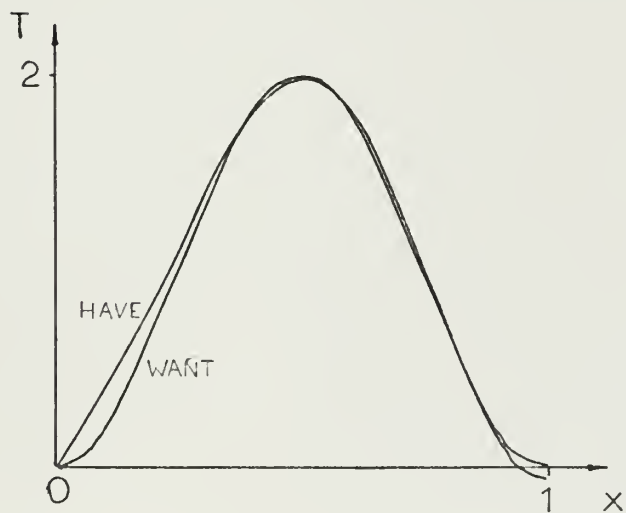
The nonlinearity was removed and it was decided to reduce the deviation between the theoretic and real outputs in the central zone. In order to achieve this result a greater weight was given to the differences between positions eight and fourteen, namely ten times for all except the central one (number eleven) which received a weight of one hundred. As a result the square error is not minimized but the distribution approaches the desired one.



a) $T = 0.0025$



b) $T = 0.005$



c) $T = 0.1$

Figure 6.3. Run No. 9

The resultant set of τ 's was:

$$\text{TAU}(1) = 116.9$$

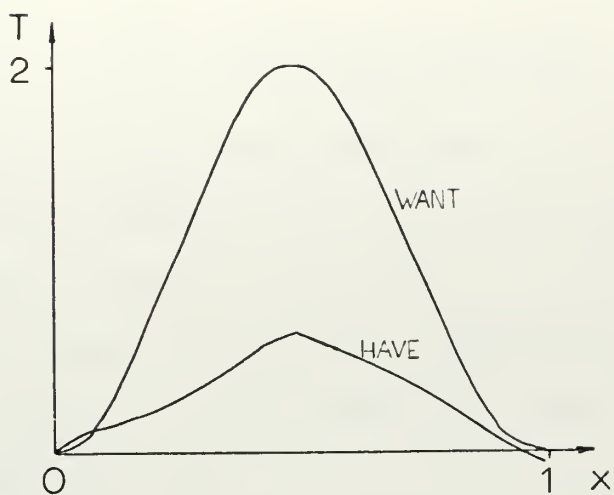
$$\text{TAU}(2) = 88.2$$

$$\text{TAU}(3) = -35.6$$

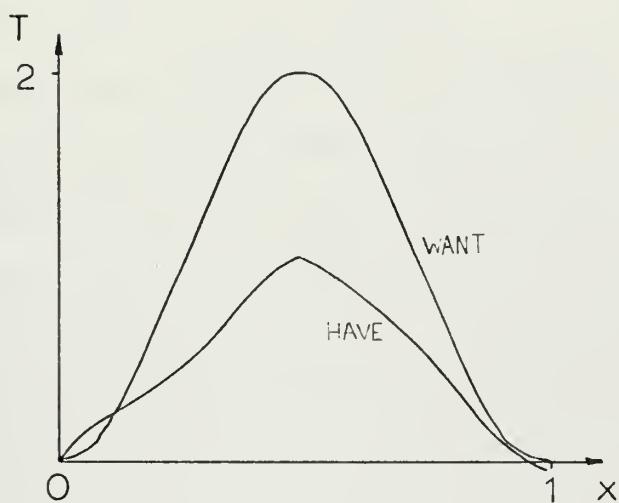
The consequences of these results may be quite important. Although in the present problem three eigenfunctions approximate already closely the required output, in other circumstances this may not happen. Then, the possibility of being able to get a closer output in some regions may indeed be relevant.

3. Run No. 10: A different Manipulator Control

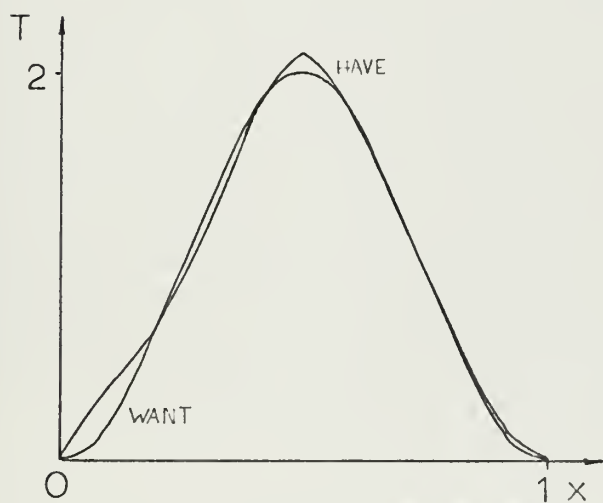
The computer program for this case is the one given as reference. As mentioned in the beginning of this section this is a situation for which the use of the compensated coefficients is not recommended. As it can be seen in the program the coefficients obtained using the gradient search came out substantially different from the theoretical ones.



a) $T = 0.0025$



b) $T = 0.005$



c) $T = 0.1$

Figure 6.4. Run No. 10

VII. CONCLUDING REMARKS

A. SUGGESTIONS FOR FUTURE RESEARCH

Many more difficult problems than the one worked out may appear, such as cases involving:

- i) Non-homogeneous boundary conditions; a transformation of variable is very useful in cases like these.
- ii) Discontinuities.
- iii) Time varying coefficients.
- iv) Hyperbolic differential equations. In this case, if using the Bubnov-Galerkin transformation the set of state variables must be augmented with those correspondent to the second derivative. Such procedure implies the need for knowing the initial conditions of the first derivatives, which is not a normal situation, and therefore the use of an observer has to be implemented.

For the beginner that gets involved in difficult situations it may be necessary to recur to the applied mathematician. However, before that, there are three references that must be considered very carefully because of the tremendous potential of knowledge they contain: Courant and Hilbert [11], Crandall [12] and Murray-Lasso [45]. Also Foster [18], working in an optimal control fashion, develops a systematic procedure for the modeling of linear regulator problems, using the Bubnov-Galerkin transformation.

As areas for future research, problems involving nonlinearities in the control, time variant coefficients and parameter identification are recommended, as well as a generalization of the Foster optimal control method for a class of problems other than the linear regulator.

B. CONCLUSIONS

The most significant of the original contributions contained in the present work is, certainly, the first closed-loop simulation of the modal control of a distributed parameter system by non-optimal control techniques. The conclusions may be summarized as follows:

- i) The classical control methods can be applied to each independent loop. A constant in the feedback is generally good enough to obtain faster response. If it is desired to work close to the stability limit, design procedures such as Bode diagrams or root-locus plots also seem to be very suitable to this purpose.
- ii) The available theory states that whenever the control is linear (at least in steady-state), the Fourier Coefficients of the desired output distribution, multiplied by the factor $\frac{1 + \phi_j \lambda_j G_j}{\lambda_j G_j}$, give the compensated Fourier coefficients which used as forcing functions (τ) insure that the mean square error of the output is minimized.

As shown in the computer results the above statement was proven to be not always true, specifically when the transformations \underline{Q} or \underline{P} are not accurate, which is the most general situation. Notice however that \underline{Q} is an

exact transformation (identity matrix) for the case of eigenfunction control. Also \tilde{P} and \tilde{Q} may be quite inaccurate when they are low order matrices and it is necessary to use a large number of eigenfunctions to approximate with precision the output or the control. When this happens the compensated coefficients may be quite far from optimal; such situation can be partially avoided using the technique of gradient search for the reference coefficients τ , as originally developed in this thesis.

- iii) Through an adequate definition of a cost function it is possible to vary the reference coefficients in such a way that the minimum square error is always achieved or even such that some regions fit the required distribution more closely than that minimum square error approximation.
- iv) The best set of manipulators is the one that gives a heat distribution coincident with the eigenfunctions. In this case \tilde{G} becomes the identity matrix and the system can be easily modeled using the Bubnov-Galerkin method. In many circumstances it will be difficult to have that type of manipulators; then, the set that approximates more closely that distribution is the desired one.

APPENDIX A

TRANSFORMATION OF THE PARTIAL DIFFERENTIAL EQUATION INTO AN ORDINARY DIFFERENTIAL EQUATION BY THE BUBNOV-GALERKIN METHOD

The Bubnov-Galerkin method much used by Foster [Ref. 17], and described in great detail by Mikhlin and Smolitskiy [Ref. 43], is an approximation technique for the solution of the equation $N y - q = 0$, where N is a differential operator ($\frac{\partial}{\partial t} - L_x$ in this case) that must obey certain conditions. The conditions are boundedness and existence of a completely continuous inverse, and they will be satisfied if $y(x, t)$ is unique and the coefficients of N , as well as their first derivatives are continuous.

Although apparently limited, the technique applies to a large variety of physical situations and it seems to start playing a very important role in the control of D.P.S.'s. It is a generalized and systematic way of getting the uncoupling effect obtained in Chapter IV.

Start by writing an approximate solution of the partial differential equation in terms of the eigenfunctions:

$$y(x, t) = \sum_{n=1}^{\infty} \omega_n(t) u_n(x) \approx \sum_{n=1}^N \omega_n(t) u_n(x) \quad {}^{18} \quad (A.1)$$

¹⁸ In the simulation run No. 7 vector ω is represented by vector \tilde{z} .

Applying the condition that $Ny - q$ is orthogonal to the N eigenfunctions considered here it follows that

$$\frac{d\omega(t)}{dt} = A\omega(t) + \zeta(t) \quad (A.2)$$

where

$$\begin{aligned} \omega(t) &= [\omega_1(t) \ \omega_2(t) \ \dots \ \omega_N(t)]^T \\ \zeta(t) &= [\zeta_1(t) \ \zeta_2(t) \ \dots \ \zeta_N(t)]^T \\ \zeta_i(t) &= \langle q(x,t), u_i(x) \rangle \\ [A]_{ij} &= \langle u_i(x), L_x u_j(x) \rangle \end{aligned} \quad (A.3)$$

1. Solving for $\zeta_i(t)$

$$\zeta_i(t) = \int_0^1 q(x,t) u_i(x) dx \quad (A.4)$$

but,

$$q(x,t) = \sum_{m=1}^M \alpha_m(t) H_m(x, \varepsilon_m) \quad (A.5)$$

where α_m and H_m were defined in Eqs. 4.56 and 4.51. Substituting A.5 in A.4 it follows

$$\zeta_i(t) = \sum_{m=1}^M b_{mi} \alpha_m(t) \quad (A.6)$$

with

$$b_{mi} = \int_0^1 u_i(x) H_m(x, \varepsilon_m) dx \quad (A.7)$$

which is the same as the previous definition of the elements of B .

Therefore Eq. A.2 may now be rewritten as

$$\frac{d\omega(t)}{dt} = A\omega(t) + B^T \alpha(t) \quad (A.8)$$

In the case of N filters and manipulators $\alpha(t) = (\tilde{B}^T)^{-1} \underline{\mu}$ and $\tilde{B}^T \alpha(t) = \tilde{B}^T (\tilde{B}^T)^{-1} \underline{\mu} = \underline{\mu}$ which means that it is not necessary to consider the matrix \tilde{B}^T in the model. As a consequence, the simulation by this method only reproduces the real situation when the heat distribution of the manipulators coincides with the eigenfunctions, giving $\tilde{B} = \underline{I}$.

2. Solving for $[A]_{ij}$

$$[A]_{ij} = \int_0^1 u_i(x) \cdot \frac{\partial^2 u_j}{\partial x^2} dx \quad (A.9)$$

and, from Chapter V, $u_i(x,s) = \sqrt{2} \sin \sqrt{\lambda-s} x$. Because M and M^+ are self-adjoints, the normalized eigenfunction set is orthogonal and applying the inherent properties the matrix A becomes diagonal.

A parenthesis is opened to say that the computation of the eigenfunctions done in Chapter V.B was not indispensable. As a matter of fact it would have been enough to choose an arbitrary orthogonal basis forming a complete set and satisfying the boundary conditions, and do all the derivations with this basis. One set that is chosen very often and that, by coincidence, is the derived set of eigenfunctions is

$$\beta_i(x) = \sqrt{2} \sin \frac{2i-1}{2} \pi x. \quad (A.10)$$

Back to the elements of \tilde{A} their values are:

$$\begin{aligned} A_{11} &= -\frac{\pi^2}{2} \int_0^1 \sin^2 \frac{\pi x}{2} dx = -\frac{\pi^2}{4} = -2.46740 \\ A_{22} &= -\frac{9\pi^2}{2} \int_0^1 \sin^2 \frac{3\pi x}{2} dx = -\frac{9\pi^2}{4} = -22.20661 \\ A_{33} &= -\frac{25\pi^2}{2} \int_0^1 \sin^2 \frac{5\pi x}{2} dx = -\frac{25\pi^2}{4} = -61.68503 \\ A_{ij} &= 0, \quad i \neq j \end{aligned} \quad (A.11)$$

and now all the elements in Eq. A.8 are known. The initial conditions are $\zeta_i(0)$, equal to the i^{th} coefficient of the expansion of the initial state $y(x,0)$ and for the specific problem treated here they turn out to be zero.

The output is given by

$$y(x,t) = \tilde{C}(x)\tilde{\omega}(t) \quad (\text{A.12})$$

where

$$\tilde{C}(x) = [u_1(x) \ u_2(x) \ \dots \ u_N(x)] \quad (\text{A.13})$$

APPENDIX B

BASIC DEFINITIONS IN FUNCTIONAL ANALYSIS

1. Operator

An operator is a mapping of functions into functions.

Figure B.1 is a block diagram representation of the operator L , which maps the function $m(x,t)$ into the function $y(x,t)$. The set $\{m\}$ for which L is defined is the domain of L . The corresponding set $\{y\}$ is the range of L . An operator can take several configurations, the most frequent of which are the differential operator and its inverse, the integral operator.

When computing the inverse of an operator ($m(x,t) = L^{-1}y(x,t)$) it is necessary to know the points at which it is singular and these points are called the eigenvalues λ_i , of the direct operator (L). The eigenfunctions are then defined as the functions u which satisfy the equality

$$L^{-1}u = M u = \lambda u \quad (B.1)$$

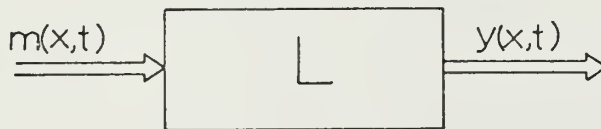


Figure B.1. Representation of the Uncontrolled System in Operator Form

The partial differential equations considered here have an infinite complete set of eigenfunctions and by complete, it is meant that any function with a finite number of discontinuities can be represented by a linear combination of these eigenfunctions. This is the concept of generalized Fourier series.

2. Inner Product

The inner product of two continuous infinite vectors p and q is defined as

$$\langle p, q \rangle = \int_a^b p \, q \, dx \quad (\text{B.2})$$

for finite values of the integral. If the vectors turn out to be discrete infinite this becomes

$$\langle p, q \rangle = \lim_{N \rightarrow \infty} \sum_{i=1}^N p_i q_i \quad (\text{B.3})$$

provided that also this summation is finite. The limit is dropped for the case of finite vectors.

When two sets of vectors are such that

$$\begin{aligned} \langle p, q \rangle &= 0, \quad p \neq q \\ &= 1, \quad p = q \end{aligned} \quad (\text{B.4})$$

these two sets are said to be orthonormal.

3. Adjoint Operator

The adjoint of an operator M is the operator M^+ such that

$$\langle M u, v \rangle = \langle u, M^+ v \rangle \quad (\text{B.5})$$

An important property of the adjoint operator is that its eigenfunctions (v_i) form a set orthogonal to the eigenfunction's set (u_i) of M .

If $M = M^+$ the operator is said to be self-adjoint and in this case both M and M^+ have the same orthogonal set of eigenfunctions. Its properties are analogous to the properties of the symmetric matrix and it can be proved that the self-adjoint case gives real eigenvalues.

4. Spectral Representation of an Operator

Using the orthogonality of the sets u_i and v_i , previously normalized, it follows easily that any operator with a discrete infinite spectrum may be represented as

$$L[\cdot] = \int_a^b \left\{ \sum_{i=1}^{\infty} \lambda_i u_i(x) v_i(\zeta) \right\} [\cdot] d\zeta \quad (B.6)$$

which takes the name of spectral representation of an operator.

If $q(x) = Lp(x)$, from the last equation and also from the Fourier series for $p(x) = \sum_{i=1}^{\infty} a_i u_i(x)$ and $q(x) = \sum_{i=1}^{\infty} b_i u_i(x)$ it turns out that

$$q(x) = \sum_{i=1}^{\infty} a_i L[u_i(x)] = \sum_{i=1}^{\infty} \lambda_i a_i u_i(x) \quad (B.7)$$

Therefore

$$\sum_{i=1}^{\infty} b_i u_i(x) = \sum_{i=1}^{\infty} \lambda_i a_i u_i(x) \quad (B.8)$$

from which, by the orthonormality properties,

$$b_j = \lambda_j a_j \quad (B.9)$$

This means that the coefficients of the eigenfunction expansion of the output can simply be obtained from the corresponding coefficients of the input by a simple product and therefore the operator was diagonalized.

A useful property of the diagonalized operator is the function of an operator which simply states

$$f(L[\cdot]) = \int_a^b \left(\sum_{i=1}^{\infty} f(\lambda_i) u_i(x) v_i(\zeta) \right) [\cdot] d\zeta \quad (B.10)$$

For the important case of the inverse operator the spectral representation is

$$L^{-1}[\cdot] = \int_a^b \left\{ \sum_{i=1}^{\infty} \frac{1}{\lambda_i} u_i(x) v_i(\zeta) \right\} [\cdot] d\zeta \quad (B.11)$$

In Ref. 45 and extensive theory of operators is given, which furnishes a deep knowledge about procedures for dealing with complex situations. This thesis will be restricted to separable operators for which in the case of partial differential operators of the type

$$L_{x,t}[m(x,t)] = H_t[m] + H_x[m] \quad (B.12)$$

The separability is realized if

$$H_t[H_x[\cdot]] = H_x[H_t[\cdot]] \quad (B.13)$$

Further restricting H_t to have constant coefficients and Laplace transforming the time, the spectral representation of L becomes

$$L_{x,s}[\cdot] = \sum_{i=1}^{\infty} \lambda_i(s) u_i(x) \int_0^1 v_i(\zeta) d\zeta \quad (B.14)$$

A more systematic way of obtaining the diagonalization of an operator is using the Bubnov-Galerkin transformation, as described in Appendix A.

5. Other Definitions

Linear operator: an operator L is linear if the mapping that it implies is such that for arbitrary scalars \underline{a} and \underline{b} it follows that $L(ax_1 + bx_2) = aLx_1 + bLx_2$.

Continuous operator: a continuous operator is characterized by the fact that if a sequence of vectors $\{x_n\}$ converges to x , then the sequence of vectors $\{Lx_n\}$ converges to Lx .

Completely continuous operator: a linear operator is completely continuous if for every bounded sequence $\{x_n\}$ in a linear normed space X the set $\{Lx_n\}$ has a subsequence which converges in the mean to an element of X .

Convergence in the Mean, also called strong convergence: the sequence $\{x_n\}$ is said to converge in the mean to x when

$\|x - x_n\| \rightarrow 0$, where $\|x\| = \sqrt{\int_a^b x^2(\zeta) d\zeta} < \infty$ (this is the case of a

L^2 space, which implies that x is a function defined in the

interval $a-b$) or $\|x\| = \sqrt{\sum_{i=1}^{\infty} |x_i^2|} < \infty$ (for $\{x\}$ a set of complex

numbers and it is said that the space is an l^2 space).

Bounded operator: a linear operator L is bounded if its domain is the entire space and if there is a scalar M such that $\|Lx\| \leq M \|x\|$. The smallest of these bounds is called norm of the operator and denoted by $\|M\|$.

Subsequence: given a sequence $S: \{Lx_1, Lx_2, Lx_3, \dots\}$ a subsequence of S is a fraction of it given by $Lx_{n_1}, Lx_{n_2}, Lx_{n_3}, \dots, Lx_{n_i}, \dots$ where $n_1 < n_2 < n_3 < \dots < n_i < \dots$.

BROAD COMPUTATION OF THE OPTIMAL POSITIONS FOR THE SENSORS.
THIS PROGRAM DOESN'T TAKE INTO ACCOUNT THE POSSIBILITY OF ONE SENSOR AT X=0.0 BECAUSE THE TEMPERATURE IS KNOWN AT THIS POINT FROM THE BOUNDARY CONDITIONS.

THE MAIN PROGRAM MUST BE ADJUSTED ACCORDING TO THE NUMBER OF SENSORS. HOWEVER, THE SUBPROGRAM USED (EVMAX) IS VERY GENERAL AND REQUIRES ONLY THE ADJUSTMENT OF THE DIMENSIONS AND DATA AND OF THE LOOP "DO 10 ---".
EVMAX DOES NOT HOLD IF MATRIX A IS COMPLEX

DIMENSION X(3),Y(3)
EXTERNAL EVMAX

N= NUMBER OF MEASUREMENTS
NG= NUMBER OF DIVISIONS

```
DATA N/3/,NG/10/
Z=.1E25
1 NG1=NG-2
  NG2=NG-1
  NG3=NG
  N1=1
  N2=N1
  DO 2 I1=N1,NG1
    N2=N2+1
    N3=N2
    DO 2 I2=N2,NG2
      N3=N3+1
      DO 2 I3=N3,NG3
        X(1)=(.1)*I1
        X(2)=(.1)*I2
        X(3)=(.1)*I3
        IF(EVMAX(X).LT.Z)GO TO 3
      GO TO 2
    3 DO 4 I=1,N
      4 Y(I)=X(I)
      Z=EVMAX(X)
    2 CONTINUE
    7 WRITE (6,5)
```

NEXT THREE STATEMENTS MUST BE ADJUSTED FOR N.GT.3

```
WRITE (6,6) Z,(Y(I),I=1,N)
5 FORMAT ('1',/////////,T35,'EVMAX',T49,'X1',T64,'X2',
1T79,'X3',/////)
6 FORMAT (' ',T33,E10.3,T42,3(F10.3,5X))
STOP
END
```

FUNCTION EVMAX(X)

THE DIMENSION MUST BE ADJUSTED IN THE NEXT STATEMENT ACCORDING TO THE NUMBER OF EIGENVALUES THAT APPROXIMATE THE SYSTEM.
ALSO THE DATA MUST BE CHANGED IF A DIFFERENT SYSTEM IS USED.
EVMAX DOESN'T HOLD IF MATRIX "A" IS COMPLEX.

C
C

```

      DIMENSION A(3),X(3),U(3,3),UI(3,3),L(3),M(3),UIT(3,3),
1 EVABS(3),UIUIT(3,3),EIVU(3)
      DATA N/3/,A/1.57080,4.71239,7.85398/
      DO 10 J=1,N
      DO 10 I=1,N
      B=A(J)*X(I)
      U(I,J)=1.41421*SIN(B)
10  UI(I,J)=U(I,J)
      DO 15 I=1,N
15  WRITE(6,300) I,X(I)
      CALL MINV(UI,N,D,L,M)
      IF(D.EQ.0.0)GO TO 50
      DO 30 J=1,N
      DO 30 I=1,N
30  UIT(J,I)=UI(I,J)
      CALL MPRD(UIT,UI,UIUIT,N,N,0,0,N)
      DO 20 I=1,N
      DO 20 J=1,N
20  WRITE (6,200) I,J,UIUIT(I,J)

```

C
C
C
C
C
C
C
C
C
C

S/R JACVAT COMPUTES THE EIGENVALUES OF A REAL
SYMMETRIC MATRIX,WHICH IS OUR CASE,BECAUSE UIUIT
IS THE PRODUCT OF MATRIX "UIT" AND ITS TRANSPOSE
"UI"

CALL JACVAT (UIUIT,N,0,EIVU,DUMMY,N)

EIVU IS THE VECTOR OF COMPUTED EIGENVALUES AND
EVMAX IS THE MAXIMUM EIGENVALUE

```

      DO 25 I=1,N
25  WRITE (6,400) I,EIVU(I)
      DO 40 I=1,N
      EVABS(I)=ABS(EIVU(I))
40  CONTINUE
      EVMAX=-1.0
      DO 45 I=1,N
      IF(EVABS(I).GT.EVMAX) GO TO 46
      EVMAX=EVABS(I)
      GO TO 45
46  EVMAX=EVABS(I)
45  CONTINUE
      GO TO 60
50  EVMAX=.1E20
200 FORMAT ('0','UIUIT(',I2,',',I2,',')=' ',D15.5)
300 FORMAT ('0','X(',I2,',')=' ',F10.5)
400 FORMAT ('0','EIVU(',I2,',')=' ',E14.5)
60  RETURN
      END

```


----- RIGOROUS COMPUTATION OF THE SENSORS -----
 ----- POSITIONS BY GRADIENT SEARCH -----

THE MAIN PROGRAM AND THE SUBPROGRAM EVMAX
 ARE VERY GENERAL AND THE ONLY THING THAT NEEDS
 TO BE CHANGED ACCORDING TO THE SYSTEM ARE THE
 DIMENSIONS, THE DATA, LOOP "DO 10 ---" AND, IF
 NECESSARY, THE PARAMETERS OF S/R DIRECT.
 EVMAX DOESN'T HOLD IF MATRIX "A" IS COMPLEX.

```

DIMENSION X(3)
EXTERNAL EVMAX
DATA N/3/,X/.3,.6,.9/
DELCAP=.01
CALL DIRECT (X,N,EVMIN,DELCAP,.25,1.E-4,EVMAX,KONVRG,
10,1)
WRITE (6,300)
WRITE (6,400) KONVRG
WRITE (6,500)
DO 10 I=1,N
10 WRITE (6,600) X(I)
300 FORMAT ('1',T60,'KONVRG')
400 FORMAT ('0',T60,I4,////)
500 FORMAT ('0',T60,'X')
600 FORMAT ('0',T50,F15.5)
STOP
END
  
```

FUNCTION EVMAX(X)

THE DIMENSION MUST BE ADJUSTED IN THE NEXT THREE
 CARDS ACCORDING TO THE NUMBER OF EIGENVALUES
 THAT APPROXIMATE THE SYSTEM.
 ALSO THE DATA MUST BE CHANGED IF A DIFFERENT
 SYSTEM IS USED.

```

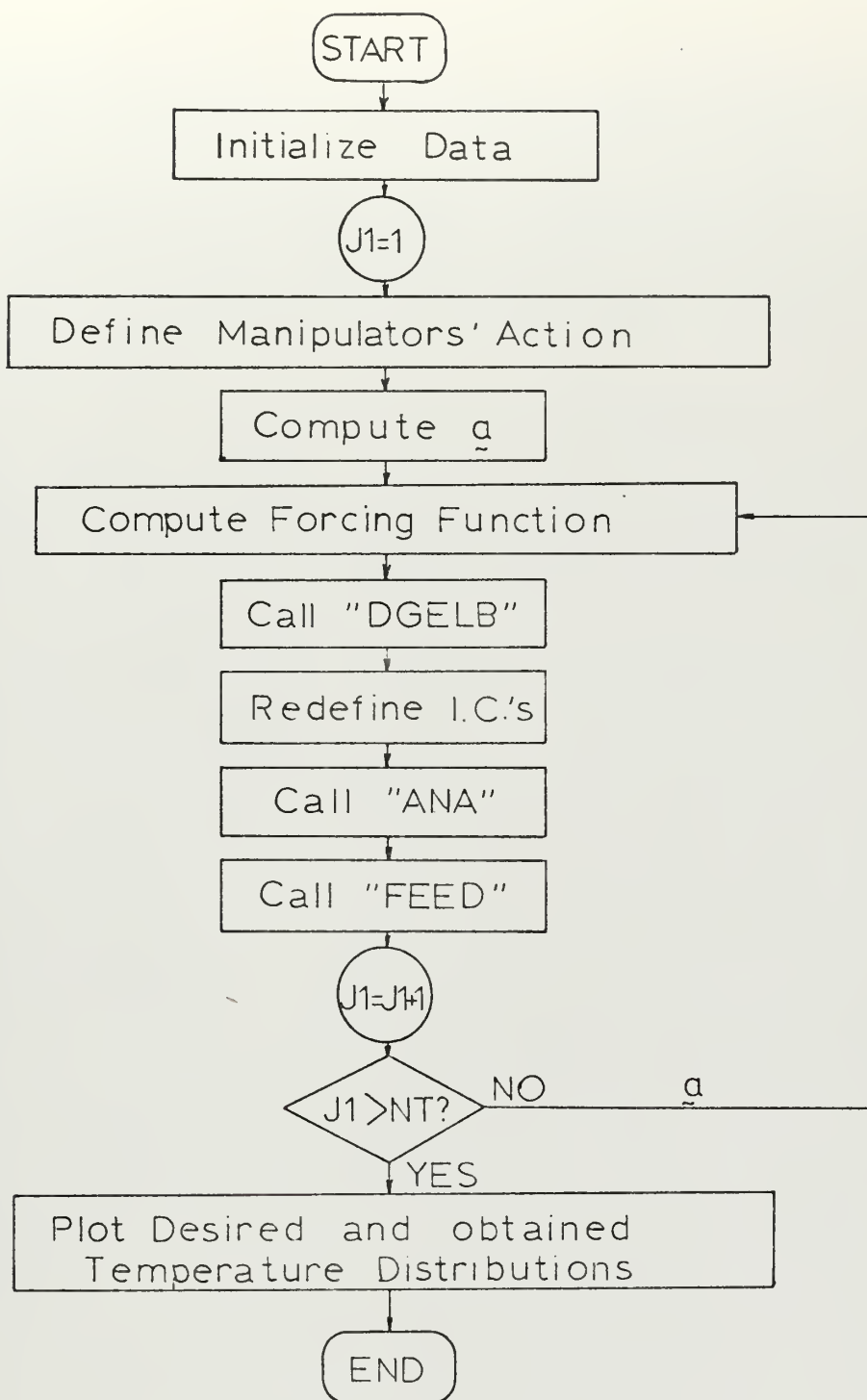
DIMENSION A(3),X(3),U(3,3),UI(3,3),L(3),M(3),UIT(3,3),
1 EVABS(3),UIUIT(3,3),EIVU(3)
DATA N/3/,A/1.57080,4.71239,7.85398/
DO 10 J=1,N
DO 10 I=1,N
B=A(J)*X(I)
U(I,J)=1.41421*SIN(B)
10 UI(I,J)=U(I,J)
DO 15 I=1,N
15 WRITE(6,300) I,X(I)
CALL MINV(UI,N,D,L,M)
IF(D.EQ.0.0)GO TO 50
DO 30 J=1,N
DO 30 I=1,N
30 UIT(J,I)=UI(I,J)
CALL MPRD(UIT,UI,UIUIT,N,N,0,0,N)
DO 20 I=1,N
DO 20 J=1,N
20 WRITE (6,200) I,J,UIUIT(I,J)
  
```

S/R JACVAT COMPUTES THE EIGENVALUES OF A REAL
 SYMMETRIC MATRIX, WHICH IS OUR CASE, BECAUSE UIUIT
 IS THE PRODUCT OF MATRIX "UIT" AND ITS TRANSPOSE
 "UI"


```

C      CALL JACVAT (UIUIT,N,0,EIVU,DUMMY,N)
C
C      EIVU IS THE VECTOR OF COMPUTED EIGENVALUES AND
C      EVMAX IS THE MAXIMUM EIGENVALUE
C
DO 25 I=1,N
25 WRITE (6,400) I,EIVU(I)
DO 40 I=1,N
EVABS(I)=ABS(EIVU(I))
40 CONTINUE
EVMAX=-1.0
DO 45 I=1,N
IF(EVABS(I).GT.EVMAX) GO TO 46
EVMAX=EVABS(I)
GO TO 45
46 EVMAX=EVABS(I)
45 CONTINUE
GO TO 60
50 EVMAX=.1E20
200 FORMAT ('0','UIUIT(',I2,',',I2,',')=',D15.5)
300 FORMAT ('0','X(',I2,',')=',F10.5)
400 FORMAT ('0','EIVU(',I2,',')=',E14.5)
60 RETURN
END

```

Flow Graph for the Solution
of the P.D.E. Using Crank-Nicholson Method

MODAL CONTROL OF A DISTRIBUTED PARAMETER SYSTEM
DESCRIBED BY A PARABOLIC DIFFERENTIAL EQUATION.

CARDS UP TO "DO 4--" AND THE DATA CARDS MUST BE
CHANGED ACCORDING TO WHAT IS REQUIRED FROM THE
SYSTEM AND ALSO ACCORDING TO THE PHYSICAL LIMITA-
TIONS. THE ONLY ADDITIONAL CHANGES MAY BE IN LOOP
"DO 3R--", WHICH DEFINES THE HEAT DISTRIBUTION OF
THE MANIPULATORS, IN STATEMENT #43, IN THE LOOP
"DO 41--" AND IN THE PARAMETERS AND VECTOR Y2 OF
S/R "DRAW".

```
REAL*8 R,A1,ITITL1(12),ITITL2(12),ITITL3(12)
REAL*4 MIU
REAL LABEL1/4HWANT/
REAL LABEL2/4PHAVE/
DIMENSION ALPHA(3),U(21,40),R(20),A(59),Q(3,3),
1P(3,3),X(3),EV1(21),EV2(21),EV3(21),H1(21),H2(21),
2H3(21),TAU(3),TEMP(3),X1(21),Y1(21),PHI(3),A1(59),
3R1(80),R2(20),MIU(3),OMEGA(3),RO(3),Y2(21)
```

```
N----# OF EQUATIONS
M----# OF RIGHT HAND VECTORS IN THE SYSTEM OF EQUA-
TIONS TO BE SOLVED BY CRANK-NICOLSON METHOD.
KF----# OF FILTERS
KS----# OF SENSORS
MP----# OF MANIPULATORS
```

```
DATA N,M,KF,KS,MP,NT/20,4,3,3,3,40 /
X(1)=.28570
X(2)=.57144
X(3)=.85715
TAU(1)=102.86
TAU(2)=62.15
TAU(3)=-60.61
PHI(1)=118.4353
PHI(2)=98.6961
PHI(3)=59.2177
```

READING THE MATRICES P AND Q OF THE FEEDBACK LOOP

```
DO 4 I=1,KF
4 READ (5,2000) (P(I,J),J=1,KS)
DO 5 I=1,MP
5 READ (5,2000) (Q(I,J),J=1,KF)
N1=N+1
N2=N+2
H=1.0/M
HK=H**2
J1=1
```

BOUNDARY CONDITIONS AT X=0

```
DO 2 J=1,NT
2 U(1,J)=0.0
```

INITIAL CONDITIONS

```
DO 3 I=1,N1
3 U(I,1)=0.0
```

SOLUTION BY CRANK-NICHOLSON METHOD

DEFINING THE TRIDIAGONAL MATRIX OF EQUATIONS ACCOR-
DING TO THE REQUIREMENTS OF S/R DGELB.

```
NM=N*(M-1)
```



```

NMO=NM-2
NO=NM-3
DO 30 I=1,NMO,3
A(I)=4.0
IF(I.EQ.NMO) GO TO 30
I1=I+1
I2=I+2
A(I1)=-1.0
A(I2)=-1.0
30 CONTINUE
A(NO)=-2.0

C
C
C   DEFINING THE FIRST (M-1)*N TERMS OF R1
C
DO 35 IA=1,NM
35 R1(IA)=0.0

C
C
C   HEAT DISTRIBUTION OF THE MANIPULATORS.THE NEXT CARDS,
C   UP TO STATEMENT #38 MUST BE CHANGED IF THE DISTRIBUTION OF
C   HEAT CORRESPONDENT TO EACH MANIPULATOR ALSO
C   CHANGES.
C
DO 38 ID=1,N1
EV1(ID)=6.283186*H*(ID-1)
IF(ID.GE.11)EV1(ID)=0.
EV2(ID)=3.141593*H*(ID-1)
EV3(ID)=3.141593*H*(ID-1)
IF (ID.LE.11) EV3(ID)=0.
H1(ID)=SIN(EV1(ID))
H2(ID)=SIN(EV2(ID))
38 H3(ID)=-COS(EV3(ID))

C
C
C   IF I.C.'S.NE.0 INSTEAD OF TAU WE MUST COMPUTE AND USE
C   MIU.
C
CALL MPRD(Q,TAU,ALPHA,MP,KF,0,0,1)
WRITE (6,1000) ALPHA
40 CONTINUE
42 DO 43 I=1,N
ID=I+1
IE=NM+I
43 R1(IE)=(ALPHA(1)*H1(ID)+ALPHA(2)*H2(ID)+ALPHA(3)*
1H3(ID))*HK*2.0
WRITE(6,1000) ALPHA

C
C
C   S/R DGELB SOLVES IN DOUBLE PRECISION A SYSTEM OF
C   LINEAR EQUATIONS.
C
DO 44 I=1,NMO
44 A1(I)=A(I)
N3=2*N+1
N4=3*N+1
DO 45 I=1,N
R2(I)=R1(I)+R1(N1)+R1(N3)+R1(N4)
N1=N1+1
N3=N3+1
45 N4=N4+1
DO 47 I=1,N
R(I)=R2(I)
47 CONTINUE
MUD=1
MLD=1
CALL DGELB (R,A1,N,1,MUD,MLD,.1E-06,IER1)
WRITE(6,8000) IER1

C
C
C   REDEFINING THE I.C.'S
C
DO 48 ID=2,N
48 R1(ID)=0.0
NE=2*N-1
NO=N-1
NE1=NE+1

```



```

      IG=1
      DO 49 ID=N2,NE
      R1(ID)=R(IG)
49.  IG=IG+1
      R1(NE1)=2*R(NO)
      NF=3*N
      NE2=NE+2
      IG=2
      DO 50 ID=NE2,NF
      R1(ID)=R(IG)
50.  IG=IG+1
      R1(NF)=0.0
      N1=N+1
      DO 60 ID=2,N1
      IF=ID-1
      U(ID,J1)=R(IF)
60.  WRITE (6,3000) ID,J1,U(ID,J1)

```

C
C
C

COMPUTATION OF VECTORS ARG AND VAL FOR THE INTERPOLATING S/R "ALI".

```

      CALL ANA (TEMP,KS,J1,H,N,U,X,N1,NT)
      CALL FEED (TEMP,ALPHA,KF,KS,MP,P,Q,PHI,TAU,MIU,OMEGA,
1.  RC)
      WRITE (6,6000) J1,(ALPHA(I),I=1,MP)
      J1=J1+1
      IF(J1.GT.NT) GO TO 90
      GO TO 40
90.  DO 91 IH=1,N1
      X1(IH)=(IH-1)*H
      D=6.283185*X1(IH)
      Y2(IH)=1.0-COS(D)
91.  CONTINUE
      IQ=1
      DO 92 IH=1,N1
92.  Y1(IH)=U(IH,IQ)
      PEAD(5,4000) ITITL1
      READ(5,4000) ITITL2
      PEAD(5,4000) ITITL3
      CALL DRAW(21,X1,Y2,1,0,LABEL1,ITITL1,0,0,1,1,2,2,7,7,
11, LAST)
      CALL DRAW(21,X1,Y1,3,0,LABEL2,ITITL1,0,0,1,1,2,2,7,7,
11, LAST)
      IQ=2
      DO 93 IH=1,N1
93.  Y1(IH)=U(IH,IQ)
      CALL DRAW(21,X1,Y2,1,0,LABEL1,ITITL2,0,0,1,1,2,2,7,7,
11, LAST)
      CALL DRAW(21,X1,Y1,3,0,LABEL2,ITITL2,0,0,1,1,2,2,7,7,
11, LAST)
      IQ=40
94.  Y1(IH)=U(IH,IQ)
      CALL DRAW(21,X1,Y2,1,0,LABEL1,ITITL3,0,0,1,1,2,2,7,7,
11, LAST)
      CALL DRAW(21,X1,Y1,3,0,LABEL2,ITITL3,0,0,1,1,2,2,7,7,
11, LAST)
1000 FORMAT ('0','ALPHA=',3E15.5)
2000 FORMAT (8E10.5)
3000 FORMAT (' ','U(',I2,',',',I4,')=',E15.5)
4000 FORMAT(10A8)
6000 FORMAT ('0','ALPHA(',I6,')=',7E14.5)
8000 FORMAT ('0','IER1=',I5)
      END

```

C
C

SUBROUTINE ANA (TEMP,KS,J1,H,N,U,X,N1,NT)

THIS S/R COMPUTES THE INTERPOLATED VALUES OF THE TEMPERATURE AT THE EXACT POSITION OF THE SENSORS.

```

DIMENSION U(N1,NT),X(KS),TEMP(KS),ARG(6),VAL(6)
DO 80 IA=1,KS
  JA=1
  AP=X(IA)*N+1.0
  IAR=AP
  BR=AP-IAR
  IF(BR.GE.0.5) GO TO 60
  ARG(JA)=(IAR-1)*H
  VAL(JA)=U(IAR,J1)
  JAUX=0
  GO TO 70
60  ARG(JA)=IAR*H
  VAL(JA)=U(IAR+1,J1)
  JAUX=1
70  CONTINUE
  IF (JAUX.EQ.0) GO TO 73
  DO 71 JA=2,6,2
    ARG(JA)=(IAR-JA/2)*H
    VAL(JA)=U(IAR-JA/2+1,J1)
71  CONTINUE
  DO 72 JA=3,5,2
    ARG(JA)=(IAR+(JA-1)/2)*H
    VAL(JA)=U(IAR+(JA-1)/2+1,J1)
72  CONTINUE
  GO TO 75
73  DO 74 JA=2,6,2
    ARG(JA)=(IAR+JA/2-1)*H
    VAL(JA)=U(IAR+JA/2,J1)
74  CONTINUE
  DO 75 JA=3,5,2
    ARG(JA)=(IAR-(JA+1)/2)*H
    VAL(JA)=U(IAR-(JA+1)/2+1,J1)
75  CONTINUE
  CALL ALI (X(IA),ARG,VAL,TEMP(IA),6,.1E-03,IER2)
  WRITE (6,9000) IER2
80  CONTINUE
9000 FORMAT ('0','IER2=',I5)
  RETURN
  END

```

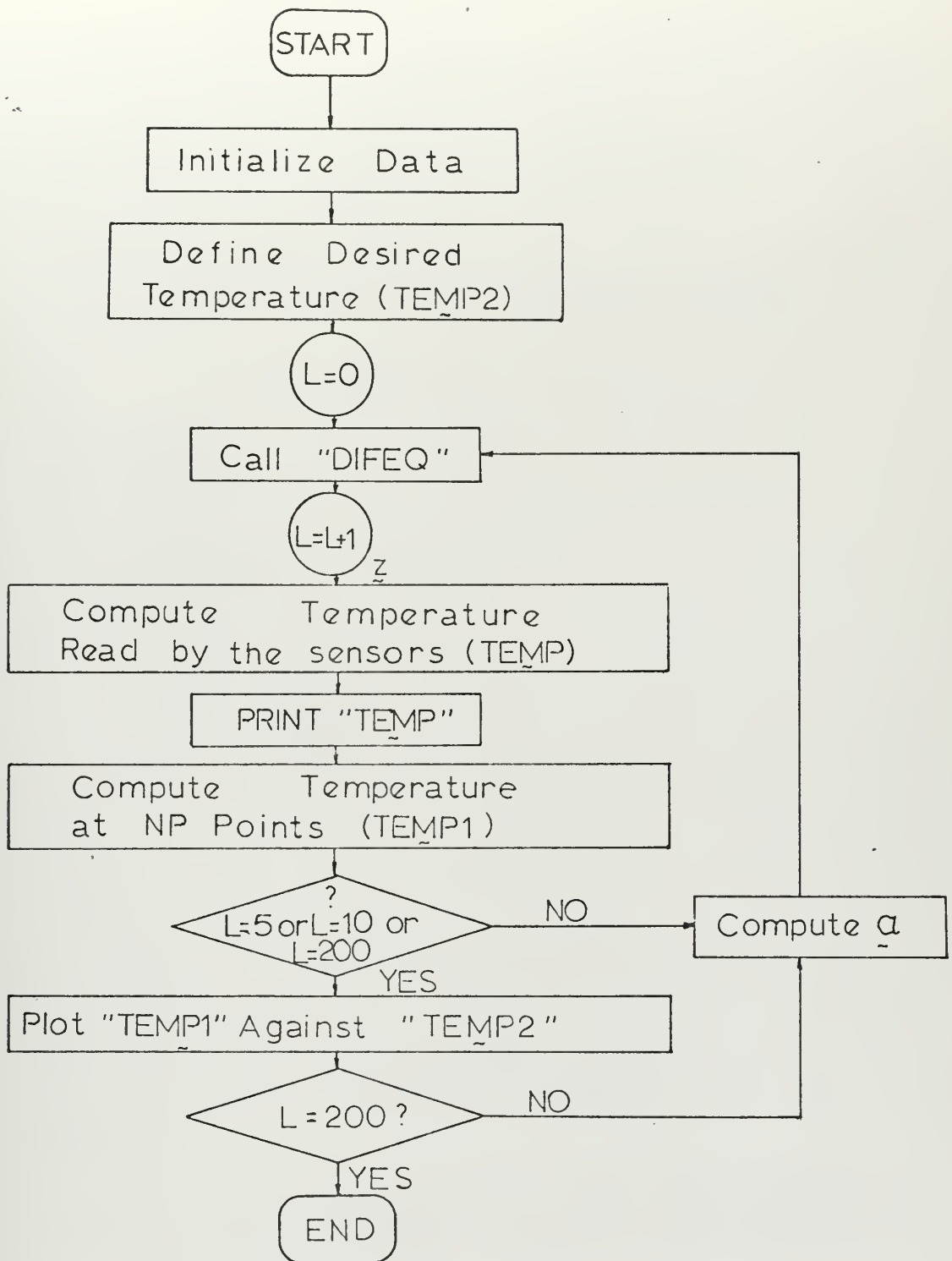
SUBROUTINE FEED (TEMP,ALPHA,KF,KS,MP,P,Q,PHI,TAU,MIU,
1 OMEGA,RO)

THIS S/R WORKS THE INFORMATION FROM THE SENSORS AND COMPUTES THE FEEDBACK CONTROL LAW.

```

REAL*4 MIU
DIMENSION Q(MP,KF),ALPHA(MP),MIU(KF),P(KF,KS),TEMP(KS)
1  OMEGA(KS),RO(KF),PHI(KF),TAU(KF)
  CALL MPRD (P,TEMP,OMEGA,KF,KS,0,0,1)
  DO 10 I=1,KF
    RO(I)=PHI(I)*OMEGA(I)
10  MIU(I)=TAU(I)-RO(I)
  CALL MPRD (Q,MIU,ALPHA,MP,KF,0,0,1)
  RETURN
  END

```

Flow Graph for the Solution
of the P.D.E. Using Bubnov-Galerkin Method


```

REAL LABEL1/4HWANT/,LABEL2/4HHAVE/,MIU
REAL*8 ITITL1(12),ITITL2(12),ITITL3(12)
DIMENSION TAU(3),Z(3),PHI(3),TEMP(3),U(3,3),
1 P(3,3),OMEGA(3),RO(3),MIU(3),R(3,3),U1(21,3),
2 TEMP1(21),TEMP2(21),A(3),X(21),Y(21),A1(3)
DATA TAU/116.1,82.9,-28.7/
DATA PHI/118.4353,98.6961,59.2177/
DATA X/0.0,.05,.1,.15,.20,.25,.30,.35,.40,.45,.50,.55,
1 .60,.65,.70,.75,.80,.85,.90,.95,1.0/,A/1.57080,
2 4.71239,7.85398/,NP/21/,A1/-2.46740,-22.20661,
3 -61.68503/
H=2.5 E-05
N=3
KS=3
KF=3
MIU(1)=TAU(1)
MIU(2)=TAU(2)
MIU(3)=TAU(3)
DO 5 J=1,N
DO 5 I=1,NP
D=A(J)*X(I)
5 U1(I,J)=1.41421*SIN(D)
DO 6 J=1,NP
E=6.283185*X(J)
6 TEMP2(J)=1.0-COS(E)
DO 10 I=1,KF
10 READ(5,2000) (P(I,J),J=1,KS)
DO 11 I=1,KF
11 READ(5,2000) (U(I,J),J=1,KS)
READ(5,4000) ITITL1
READ(5,4000) ITITL2
READ(5,4000) ITITL3
WRITE(6,5000)
L=0
Z10=0.
Z20=0.
Z30=0.
Z100=0.
Z200=0.
Z300=0.
30 CONTINUE
CALL DIFEQ(Z10,Z100,Z(1),MIU(1),A1(1),L,H)
Z100=Z10
Z10=Z(1)
CALL DIFEQ(Z20,Z200,Z(2),MIU(2),A1(2),L,H)
Z200=Z20
Z20=Z(2)
CALL DIFEQ(Z30,Z300,Z(3),MIU(3),A1(3),L,H)
Z300=Z30
Z30=Z(3)
L=L+1

IF P AND U ARE SQUARE MATRICES P*U=I AND INSTEAD OF
OMEGA WE CAN USE DIRECTLY Z. HOWEVER A SIMULATION WHERE
ERRORS IN THE MEASUREMENTS ARE TAKEN IN ACCOUNT MUST
INCLUDE ALL THE FOLLOWING STEPS.

50 CALL MPRD (U,Z,TEMP,KS,N,0,0,1)
CALL MPPD(U1,Z,TEMP1,21,3,0,0,1)
CALL MPRD (P,TEMP,OMEGA,KF,KS,0,0,1)
R=L/5.0
R1=IFIX(R)
R2=R-R1
IF(R2.LT..1E-04)GO TO 55
GO TO 56
55 WRITE(6,6000) L,TEMP1(1),TEMP1(5),TEMP1(9),TEMP1(13),
1TEMP1(17),TEMP1(21)

```



```

56 DO 60 I=1,KF
   RO(I)=PHI(I)*OMEGA(I)
   MIU(I)=TAU(I)-PO(I)
60 CONTINUE
   IF (L.EQ.100) GO TO 70
   IF (L.EQ.200) GO TO 80
   IF (L.EQ.4000) GO TO 90
   GO TO 30
70 CALL DPAW(21,X,TEMP2,1,0,LABEL1,ITITL1,0,0,1,1,2,2,
  17,7,1, LAST)
   CALL DRAW(21,X,TEMP1,3,0,LABEL2,ITITL1,0,0,1,1,2,2,
  17,7,1, LAST)
   GO TO 30
80 CALL DPAW(21,X,TEMP2,1,0,LABEL1,ITITL2,0,0,1,1,2,2,
  17,7,1, LAST)
   CALL DRAW(21,X,TEMP1,3,0,LABEL2,ITITL2,0,0,1,1,2,2,
  17,7,1, LAST)
   GO TO 30
90 CALL DRAW(21,X,TEMP2,1,0,LABEL1,ITITL3,0,0,1,1,2,2,
  17,7,1, LAST)
   CALL DRAW(21,X,TEMP1,3,0,LABEL2,ITITL3,0,0,1,1,2,2,
  17,7,1, LAST)
   GO TO 100
2000 FORMAT (8E10.5)
4000 FORMAT (10A8)
5000 FORMAT ('1',3X,'L', 8X,'TEMP1(1)',14X,'TEMP1(5)', 8X,
  1,'TEMP1(9)', 8X,'TEMP1(13)', 8X,'TEMP1(17)', 8X,
  2,'TEMP1(21)',//)
6000 FORMAT (' ',I4,E23.5,5E17.5)
100 CONTINUE
END

```

SUBROUTINE DIFEQ(Z0,Z00,Z,FIU,A,L,H)

C
C
C
C
C
C
C

THIS S/P INTEGRATES AN ORDINARY 1ST ORDER DIFFE-
RENTIAL EQUATION BY THE PREDICTION AND CORRECTION
METHOD

```

ZDOT0=A*Z00+FIU
IF (L.EQ.0) ZDOT0=0.
ZDOT1=A*Z0+FIU
I=0
L1=L+1
10 Z1=Z00+2.0*H*ZDOT1
   I=I+1
   ZDOT1=A*Z1+FIU
   Z=Z0+H/2.0*(ZDOT1+ZDOT0)
   IF (ABS(Z-Z1).LE.1.E-05) GO TO 40
   IF (I.EQ.100) GO TO 39
   GO TO 10
39 WRITE (6,1000) L1
40 RETURN
1000 FORMAT ('0','NO CONVERGENCE',3X,'L1=',I4,/)
END
//GO.SYSIN DD *
.17532 E00.31593 E00.39391 E00
.39393 E00.17533 E00-.31591E00
.31589 E00-.39391E00.17532 E00
.61357 E00.13787 E01.11058 E01
.11057 E01.61353 E00-.13788E01
.13788 E01-.11057E01.61367 E00
WANTED AND OBTAINED TEMPERATURE AT 0.0025 SEC ALM0403 A.MO
-BUBNOV-GALERKIN
WANTED AND OBTAINED TEMPERATURE AT 0.0050 SEC ALM0403 A.MO
-BUBNOV-GALERKIN
WANTED AND OBTAINED TEMPERATURE AT 0.1 SEC ALM0403 A.MO
-BUBNOV-GALERKIN

```



```

C      THIS PROGRAM IS BASICALLY THE SAME AS THE ONE THAT
C      SOLVES THE CLOSE-LOOP PROBLEM USING THE CRANK-NICOLSON
C      METHOD. THE MAIN DIFFERENCE IS IN THE FACT THAT MOST
C      OF THE WRITE STATEMENTS WERE REMOVED, AS WELL AS THE
C      PLOT OUTPUTS. BY DEFINING A COST FUNCTION WHICH IS
C      OPTIMIZED BY A GRADIENT SEARCH A SET OF TAUS IS
C      OBTAINED AND USED AS INPUT TO THE SYSTEM IN THE BASIC
C      MODEL. ONLY THEN A PLOT OUTPUT IS OBTAINED
      DIMENSION TAU(3)
      COMMON/ONE/P(3,6),Q(3,3),PHI(3),X(6)
      COMMON/TWO/N,M,KF,KS,MP,NT
      EXTERNAL SQUERR
      NW=3
      TAU(1)=116.9
      TAU(2)=88.2
      TAU(3)=-35.6
      X(1)=.15406
      X(2)=.30828
      X(3)=.46234
      X(4)=.61621
      X(5)=.76996
      X(6)=.92325

C      N----# OF EQUATIONS
C      M----# OF RIGHT HAND VECTORS IN THE SYSTEM OF EQUA_
C      TIONS TO BE SOLVED BY CRANK-NICOLSON METHOD.
C      KF----# OF FILTERS
C      KS----# OF SENSORS
C      MP----# OF MANIPULATORS
      N=20
      M=4
      KF=3
      KS=6
      MP=3
      NT=40

C      PHI(1)=118.4353
      PHI(2)=98.6961
      PHI(3)=50.2177

C      DEFINE THE MATRICES P AND Q OF THE FEEDBACK LOOP
      DO 4 I=1,KF
4      READ (5,2000) (P(I,J),J=1,KS)
      DO 5 I=1,MP
5      READ (5,2000) (Q(I,J),J=1,KF)
      DELCAP=10.
      CALL DIRECT(TAU,NW,COSMIN,DELCAP,.25,.01,SQUERR,
1      KONVRG,100,-1)
      WRITE(6,300)
      WRITE(6,400)KONVRG
      WRITE(6,500)
      DO 10 I=1,NW
10     WRITE(6,600) TAU(I)
      WRITE(6,700) COSMIN
300    FORMAT('0',T60,'KONVRG')
400    FORMAT('0',T60,I4,/)
500    FORMAT('0',T60,'TAU')
600    FORMAT('0',T50,F15.5)
700    FORMAT('0',T60,'SQUERR=',F7.3)
2000   FORMAT (8E10.5)
      STOP
      END

      FUNCTION SQUERR (TAU)

C
C
C
C
      MODAL CONTROL OF A DISTRIBUTED PARAMETER SYSTEM
      DESCRIBED BY A PARABOLIC DIFFERENTIAL EQUATION.

```


CARDS UP TO "DO 4--" AND THE DATA CARDS MUST BE CHANGED ACCORDING TO WHAT IS REQUIRED FROM THE SYSTEM AND ALSO ACCORDING TO THE PHYSICAL LIMITATIONS. THE ONLY ADDITIONAL CHANGES MAY BE IN LOOP "DO 38--", WHICH DEFINES THE HEAT DISTRIBUTION OF THE MANIPULATORS, IN STATEMENT #43, IN THE LOOP "DO 41--" AND IN THE PARAMETERS AND VECTOR Y2 OF S/R "DRAW".

```

REAL*8 R,A1
REAL*4 MIU
DIMENSION ALPHA(3),U(21,40 ),R(20),A(59 ),
1 EV1(21),EV2(21),EV3(21),H1(21),H2(21),
2 H3(21),TAU(3),TEMP(6),X1(21),Y1(21),A1(59),
3 R1(80),R2(20),MIU(3),OMEGA(3),RO(3),Y2(21),SQU(21)
COMMON/ONE/P(3,6),Q(3,3),PHI(3),X(6)
COMMON/TWO/N,M,KF,KS,MP,NT
N1=N+1
N2=N+2
H=1.0/N
HK=H**2
J1=1

```

BOUNDARY CONDITIONS AT X=0

```

DO 2 J=1,NT
2 U(1,J)=0.0

```

INITIAL CONDITIONS

```

DO 3 I=1,N1
3 U(I,1)=0.0

```

SOLUTION BY CRANK-NICHOLSON METHOD

DEFINING THE TRIDIAGONAL MATRIX OF EQUATIONS ACCORDING TO THE REQUIREMENTS OF S/R DGELB.

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NM=N*(M-1)
NM0=NM-2
NO=NM-3
DO 30 I=1,NM0,3
A(I)=4.0
IF(I.EQ.NM0) GO TO 30
I1=I+1
I2=I+2
A(I1)=-1.0
A(I2)=-1.0
30 CONTINUE
A(NO)=-2.0

```

DEFINING THE FIRST (M-1)*N TERMS OF R1

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DO 35 IA=1,NM
35 R1(IA)=0.0

```

HEAT DISTRIBUTION OF THE MANIPULATORS. THE NEXT CARDS, UP TO STATEMENT #38 MUST BE CHANGED IF THE DISTRIBUTION OF HEAT CORRESPONDENT TO EACH MANIPULATOR ALSO CHANGES.

```

DO 38 ID=1,N1
EV1(ID)=1.57080*H*(ID-1)
EV2(ID)=4.71239*H*(ID-1)
EV3(ID)=7.85398*H*(ID-1)
H1(ID)= 1.414214*SIN(EV1(ID))
H2(ID)= 1.414214*SIN(EV2(ID))
38 H3(ID)= 1.414214*SIN(EV3(ID))

```



```

C      IF I.C.'S.NE.0 INSTEAD OF TAU WE MUST COMPUTE AND USE
C      MIU.
C      CALL MPRD(Q,TAU,ALPHA,MP,KF,0,0,1)
40  CONTINUE
42  DO 43 I=1,N
      ID=I+1
      IE=NM+I
43  R1(IE)=(ALPHA(1)*H1(ID)+ALPHA(2)*H2(ID)+ALPHA(3)*
1  H3(ID))*HK*2.0
C
C      S/R DGELB SOLVES IN DOUBLE PRECISION A SYSTEM OF
C      LINEAR EQUATIONS.
C
      DO 44 I=1,NM0
44  A1(I)=A(I)
      N3=2*N+1
      N4=3*N+1
      DO 45 I=1,N
        R2(I)=R1(I)+R1(N1)+R1(N3)+R1(N4)
        N1=N1+1
        N3=N3+1
45  N4=N4+1
      DO 47 I=1,N
        R(I)=R2(I)
47  CONTINUE
      MUD=1
      MLD=1
      CALL DGELB (P,A1,N,1,MUD,MLD,.1E-06,IER1)
C
C      REDEFINING THE I.C.'S
C
      DO 48 ID=2,N
48  R1(ID)=0.0
      NE=2*N-1
      NO=N-1
      NE1=NE+1
      IG=1
      DO 49 ID=N2,NE
        R1(ID)=R(IG)
49  IG=IG+1
        R1(NE1)=2*R(NO)
        NF=3*N
        NE2=NE+2
        IG=2
        DO 50 ID=NF2,NF
          R1(ID)=R(IG)
50  IG=IG+1
          R1(NF)=0.0
          N1=N+1
          DO 60 ID=2,N1
            IE=ID-1
60  U(ID,J1)=R(IE)
C
C      COMPUTATION OF VECTORS ARG AND VAL FOR THE INTERPC_
C      LATING S/R "ALI".
C
      CALL ANA (TEMP,KS,J1,H,N,U,X,N1,NT)
      CALL FEED (TEMP,ALPHA,KF,KS,MP,P,Q,PHI,TAU,MIU,OMEGA,
1  IR0)
      J1=J1+1
      IF(J1.EQ.38) WRITE(6,6000) J1,(ALPHA(I),I=1,MP)
      IF(J1.GT.NT) GO TO 90
      GO TO 40
90  IQ=40
      WRITE (6,6000) J1,(ALPHA(I),I=1,MP)
      SQUERR=0.
      DO 95 IH=1,N1
        X1(IH)=(IH-1)*H
        D=6.283185*X1(IH)
        Y2(IH)=1.0-COS(D)
        Y1(IH)=U(IH,IQ)

```



```

      SQU(IH)=(Y1(IH)-Y2(IH))*2
95  SQUERR=SQUERR+SQU(IH)
6000 FORMAT ('0','ALPHA(',I6,')=',7E14.5)
      END

```

C
C

```

      SUBROUTINE ANA (TEMP,KS,J1,H,N,U,X,N1,NT)

```

C
C
C
C
C
C

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      THIS S/R COMPUTES THE INTERPOLATED VALUES OF THE
      TEMPERATURE AT THE EXACT POSITION OF THE SENSORS.

```

```

      DIMENSION U(N1,NT),X(KS),TEMP(KS),ARG(6),VAL(6)
      DO 80 IA=1,KS
      JA=1
      AR=X(IA)*N+1.0
      IAR=AR
      BR=AR-IAR
      IF(BR.GE.0.5) GO TO 60
      ARG(JA)=(IAR-1)*H
      VAL(JA)=U(IAR,J1)
      JAUX=0
      GO TO 70
60  ARG(JA)=IAR*H
      VAL(JA)=U(IAR+1,J1)
      JAUX=1
70  CONTINUE
      IF (JAUX.EQ.0) GO TO 73
      DO 71 JA=2,6,2
      ARG(JA)=(IAR-JA/2)*H
      VAL(JA)=U(IAR-JA/2+1,J1)
71  CONTINUE
      DO 72 JA=3,5,2
      ARG(JA)=(IAR+(JA-1)/2)*H
      VAL(JA)=U(IAR+(JA-1)/2+1,J1)
72  CONTINUE
      GO TO 75
73  DO 74 JA=2,6,2
      ARG(JA)=(IAR+JA/2-1)*H
      VAL(JA)=U(IAR+JA/2,J1)
74  CONTINUE
      DO 75 JA=3,5,2
      ARG(JA)=(IAR-(JA+1)/2)*H
      VAL(JA)=U(IAR-(JA+1)/2+1,J1)
75  CONTINUE
      CALL ALI (X(IA),ARG,VAL,TEMP(IA),6,.1E-03,IER2)
80  CONTINUE
      RETURN
      END

```

C
C

```

      SUBROUTINE FEED (TEMP,ALPHA,KF,KS,MP,P,Q,PHI,TAU,MIU,
1 OMEGA,RO)

```

C
C
C
C
C
C

```

      THIS S/R WORKS THE INFORMATION FROM THE SENSORS AND
      COMPUTES THE FEEDBACK CONTROL LAW.

```

```

      REAL*4 MIU
      DIMENSION Q(MP,KF),ALPHA(MP),MIU(KF),P(KF,KS),TEMP(KS)
1  OMEGA(KS),RO(KF),PHI(KF),TAU(KF)
      CALL MPRD (P,TEMP,OMEGA,KF,KS,0,0,1)
      DO 10 I=1,KF
      RO(I)=PHI(I)*OMEGA(I)
10  MIU(I)=TAU(I)-RO(I)
      CALL MPRD (Q,MIU,ALPHA,MP,KF,0,0,1)

```


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13. ABSTRACT <p>The present work is an attempt to put together the most relevant aspects of the engineering problems involving distributed parameter systems (D.P.S.'s). Simulation and optimal control are explained in detail in Chapters II and III.</p> <p>The original contribution of this thesis is given in Chapters V and VI, where modal control theory and a gradient subroutine that searches for the optimal reference coefficients are used. As a result, it was possible to obtain an output distribution better than the one achievable by the known methods. This technique works in situations of strongly nonlinear control and compensates the effect of having the analyzer and synthesizer approximated by low order matrices. It also makes it possible to give higher weight to some zones of the output distribution in order to have a better local fit. The necessary background for understanding Chapters V and VI is given in Chapter IV.</p>			

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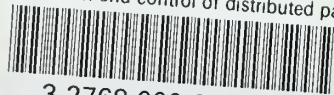
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